Mathematics

Key Stage 4

to

Key Stage 5

Transition

Students aren’t expected to complete all the work in this booklet. You should pick out areas you feel you need to improve and focus on those. There are answers at the back of the document.

ALGEBRA

CHAPTER 1

MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

Multiplication

In algebra we usually omit the multiplication sign, so that, for example

$2 \times a$ is written as $2a$
and $x \times y$ is simplified to xy

If a string of numbers and letters are multiplied, the multiplication can be done in any order; for example:

$$\begin{aligned} 2p \times 3q &= 2 \times p \times 3 \times q \\ &= 2 \times 3 \times p \times q \\ &= 6pq \end{aligned}$$

You should now attempt questions 1 to 6 of exercise 1 on page 4.

Indices can be used to simplify expressions such as $x \times x$

e.g. (a) $x \times x = x^2$
(b) $x^2 \times x = x^3$

The resulting expressions are called powers of x.
When multiplying powers of the *same* number or letter, the indices can be added.

e.g (a) $x^4 \times x = x^4 \times x^1 = x^5$
(b) $x^3 \times x^4 = x^7$

Note that $2x^2$ means only the x is squared, i.e. $2 \times x^2$, whereas $(2x)^2$ means that $2x$ is squared, i.e. $2x \times 2x = 4x^2$.

Examples

Simplify: (a) $4x^3 \times 5x^2$ (b) $3x^2y^3 \times 4x^4y$ (c) $(5x^2y)^2$

(a) $$\begin{aligned} 4x^3 \times 5x^2 &= 4 \times 5 \times x^3 \times x^2 \\ &= 20x^5 \end{aligned}$$

(b) $$\begin{aligned} 3x^2y^3 \times 4x^4y &= 3 \times 4 \times x^2 \times x^4 \times y^3 \times y \\ &= 12x^6y^4 \end{aligned}$$

(c) $$\begin{aligned} (5x^2y)^2 &= 5x^2y \times 5x^2y \\ &= 5 \times 5 \times x^2 \times x^2 \times y \times y \\ &= 25x^4 \times y^2 \end{aligned}$$

You should now attempt questions 7 to 15 of exercise 1 on page 4.

Division

The result of a division can be expressed as a fraction, for example

$$9 \div 16 = \frac{9}{16}$$

$$\text{and} \quad x \div y = \frac{x}{y}$$

Consider the expression $\frac{9 \times 16}{4}$. Instead of working out 9×16 and then dividing by 4, we can divide <u>one</u> of the terms in the numerator by 4 before multiplying:

$$\frac{9 \times 16}{4} = 9 \times \frac{16}{4} = 9 \times 4 = 36$$

Similarly

$$\frac{8x}{4} = \frac{8}{4} \times x = 2 \times x = 2x$$

and

$$\frac{12ab^2}{b^2} = 12a \times \frac{b^2}{b^2} = 12a \times 1 = 12a$$

The expression $\frac{3x}{4}$ cannot be simplified. However, it can be put into an alternative form:

$$\frac{3x}{4} = \frac{3}{4} \times x = \frac{3}{4}x$$

Similarly,

$$\frac{4xy}{5} = \frac{4}{5}xy$$

You should now attempt questions 16 to 23 on page 4

The value of the fraction is unchanged if the numerator and denominator are divided by the same quantity, for example

$$\frac{8}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$$

and

$$\frac{8x}{6y} = \frac{8x \div 2}{6y \div 2} = \frac{(8/2) \times x}{(6/2) \times y} = \frac{4x}{3y}$$

The process of dividing the numerator and denominator can be simplified by <u>cancelling</u>:

$$\frac{8x}{6y} = \frac{{}^{4}\not{8}\, x}{{}^{3}\not{6}\, y} = \frac{4x}{3y}$$

Cancelling can be carried out more than once, for example

$$\frac{9pq}{6p} = \frac{{}^{3}\not{9}\, {}^{1}\not{p}\, q}{{}^{2}\not{6}\, {}^{1}\not{p}} = \frac{3 \times 1 \times q}{2 \times 1} = \frac{3q}{2}$$

This is equivalent to dividing the numerator and denominator by 3 and then by p, which is the same as dividing by $3p$.

You should only cancel if the numerator and denominator consist of <u>products</u>.

EXERCISE 1

Simplify:

(1)	$2 \times 6x$	**(2)**	$7y \times 3$
(3)	$3p \times 4q$	**(4)**	$7a \times 9b$
(5)	$2x \times 3yz$	**(6)**	$3p \times 4q \times 5r$
(7)	$3a \times 2a$	**(8)**	$4b^2 \times 3b^3$
(9)	$3xy \times 4xz$	**(10)**	$2x^2y \times 5x^3y^2$
(11)	$(4a)^2$	**(12)**	$(3ab)^2$
(13)	$(4pq)^2 \times (3p)^2$	**(14)**	$(3x^2)^2$
(15)	$(2a^2b)^3$		

Now turn to page 2.

Simplify:

(16)	$8x \div 4$	**(17)**	$6xy \div 3$
(18)	$4x^2y \div y$	**(19)**	$5abc^2 \div c^2$

Express in an alternative form:

(20)	$\frac{2y}{3}$	**(21)**	$\frac{3ab}{4}$
(22)	$\frac{6}{7}xy$	**(23)**	$\frac{2}{3}ab^2$

Now return to page 2.

Simplify:

(24)	$8x \div 12$	**(25)**	$6 \div 9y$
(26)	$10a \div 15b$	**(27)**	$12pq \div 16p$
(28)	$18x^2y \div 3xy$	**(29)**	$20ab^3 \div 25ab^4$
(30)	$14a^3b^2 \div 21ab^5$	**(31)**	$16a^3b^2c \div 20ab^4c^2$
(32)	$(p^2q)^2 \div pq^3$		

Now start chapter 2 on page 5.

Do <u>not</u> be tempted to cancel in expressions such as :

$$\frac{4x+9y}{2x+3y}$$

<u>Examples</u>

Simplify (a) $3p^2q \div 6pq^2$ (b) $6p^2q^3c \div 9p^4q^2c^2$

Solutions: (a)

$$\begin{aligned} 3p^2q \div 6pq^2 &= \frac{3p^2q}{6pq^2} \\ &= \frac{{}^1\cancel{3} \times {}^1\cancel{p} \times p \times {}^1\cancel{q}}{{}^2\cancel{6} \times {}^1\cancel{p} \times {}^1\cancel{q} \times q} \\ &= \frac{1 \times 1 \times p \times 1}{2 \times 1 \times 1 \times q} \\ &= \frac{p}{2q} \end{aligned}$$

(b)

$$\begin{aligned} 6p^2q^3c \div 9p^4q^2c^2 &= \frac{6p^2q^3c}{9p^4q^2c^2} \\ &= \frac{{}^2\cancel{6} \times {}^1\cancel{p} \times {}^1\cancel{p} \times {}^1\cancel{q} \times {}^1\cancel{q} \times q \times {}^1\cancel{c}}{{}^3\cancel{9} \times {}^1\cancel{p} \times {}^1\cancel{p} \times p \times p \times {}^1\cancel{q} \times {}^1\cancel{q} \times {}^1\cancel{c} \times c} \\ &= \frac{2 \times q}{3 \times p \times p \times c} \\ &= \frac{2q}{3p^2c} \end{aligned}$$

You should now attempt questions 24 to 32 of exercise 1 on page 4.

CHAPTER 2

ADDITION AND SUBTRACTION OF ALGEBRAIC TERMS

The terms in an expression are the parts which are separated by plus or minus signs. For example, in the expression $4xy-5x+x^2$, the terms are: $4xy$, $5x$ and x^2.

Like terms are multiples of the same algebraic quantity.

e.g. $7x$, $5x$ and $-3x$ are three like terms; $4p^2q$ and $-2p^2q$ are two like terms.

Note that $4ab$ and $-3ba$ are like terms as ab is the same as ba.

Like terms can be added or subtracted.

e.g. (i) $$2ab+5ab=(2+5)ab$$ $$=7ab$$

(ii) $$9x^2y-2x^2y+3x^2y=(9-2+3)x^2y$$ $$=10x^2y$$

Unlike terms are multiples of different algebraic quantities; they cannot be replaced by a single term.

e.g. (i) As $2pq$ and $3pr$ are unlike terms, $2pq+3pr$ cannot be replaced by a single term.

(ii) As $3x^2$ and $4x$ are unlike terms, $3x^2-4x$ cannot be simplified.

Examples

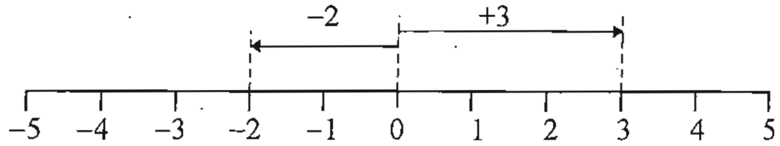
(i) $$x^2+6x+5x^2-2x-4 = x^2+5x^2+6x-2x-4$$ $$= 6x^2+4x-4$$

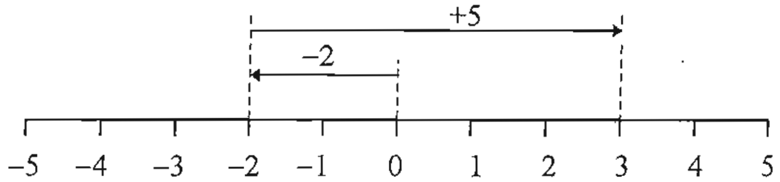
(ii) $$4a^2b+6ab^2-3ba^2+2b^2a = 4a^2b+6ab^2-3a^2b+2ab^2$$ $$= 4a^2b-3a^2b+6ab^2+2ab^2$$ $$= a^2b+8ab^2$$

You should now attempt questions 1 to 6 of exercise 2 on page 8.

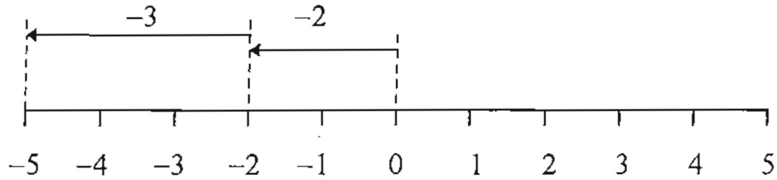
Directed Numbers

When combining like terms, it is often necessary to add or subtract directed numbers. To carry out these operations, it is helpful to represent directed numbers by arrows on a number line:





This diagram shows that $-2+5=3$



This diagram shows that $-2+(-3)=-5$

When attempting the exercise, you may find it helpful to form a mental picture of the number line.

Examples

(i) $3+(-4)+5=-1+5=4$

(ii)
$$\begin{aligned}-3ab+(-4ab)+8ab &= (-3+(-4)+8)ab \\ &= (-7+8)ab \\ &= 1ab \\ &= ab\end{aligned}$$

You should now attempt questions 7 to 12 of exercise 2 on page 8.

Any subtraction involving directed numbers can be converted into an addition.

For example,

$$4-2=4+(-2)=2$$

Subtracting 2 is the same as adding -2

In general:

To subtract a directed number, change its sign and add the resulting number.

Examples

(i) $4-6=4+(-6)=-2$

(ii) $-3-(-5)=-3+5=2$

(iii) $4-(-3)=4+3=7$

(iv) $-4-6-(-3)=-4+(-6)+3=-7$

When combining algebraic expressions it is sometimes convenient to convert an addition into a subtraction.

For example,

$4p+(-3q)$ can be written as $4p-3q$

Example

$$\begin{aligned}
a^2-3ab-4b^2-5a^2-6ab-(-8b^2) &= a^2-3ab-4b^2-5a^2-6ab+8b^2 \\
&= a^2-5a^2-3ab-6ab-4b^2+8b^2 \\
&= a^2+(-5a^2)+(-3ab)+(-6ab)+(-4b^2)+8b^2 \\
&= -4a^2+(-9ab)+4b^2 \\
&= -4a^2+(-9ab)+4b^2 \\
&= -4a^2-9ab+4b^2
\end{aligned}$$

You should now attempt questions 13 to 20 of exercise 2 on page 8.

EXERCISE 2

Simplify:

(1) $7x+11x$

(2) $16ab-4ab$

(3) $9ab^2-4ab^2-3ab^2$

(4) $4x^2+5x-2x^2+6+3x-4$

(5) $7p^2q+4pqr-3qp^2-2qpr$

(6) $a^2+2ab+4b^2+3ba+5a^2-2b^2$

Now turn to page 6.

Evaluate:

(7) $12+(-16)$

(8) $-3+(-9)$

(9) $6+(-15)+4$

Simplify:

(10) $-4x^2+5x^2+(-3x^2)$

(11) $-3a^2b+4ab^2+6a^2b+(-2ab^2)$

(12) $-3a^2+8ab+6b^2+(-4a^2)+(-3ab)+(-2b^2)$

Now turn to page 7.

Evaluate:

(13) $8-15$

(14) $-14-(-9)$

(15) $6-(-5)$

(16) $-4-5-6$

(17) $-8-(-3)-2$

Simplify:

(18) $3x^2+4x-6x^2-5x$

(19) $-6a^2+4ab+5b^2-3a^2-7ab-4b^2$

(20) $2a^2b-3ab^2-5b^3+(-5ab^2)-6a^2b-(-3b^3)$

Now turn to chapter 3 on page 9

CHAPTER 3

FURTHER MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

Laws of indices

In a numerical expression such as 2^3, the 2 is called the base and the 3 is the index (plural indices). Similarly, the base of the algebraic expression a^m is a and the index is m.

In chapter 1 we simplified expressions with indices by writing them out in full. It is quicker to apply the following laws.

LAW 1

Written out in full,

$$a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$$

i.e. $$a^2 \times a^3 = a^{2+3}$$

The general result is

$$a^m \times a^n = a^{m+n}$$

The indices can only be added if powers of the same base are added together; the law cannot be used to simplify an expression such as $a^4 b^5$.

Examples

(a) $a^6 \times a^3 = a^9$

(b) $a^3 \times a^4 \times a^5 = a^{3+4+5} = a^{12}$

(c) $2a^3 b \times 3a^4 b^3 = 6a^3 \times a^4 \times b \times b^3$
$= 6a^7 b^4$

You should now attempt questions 1 to 4 of exercise 3 on page 14.

LAW 2

Written out in full,

$$\frac{a^5}{a^2} = \frac{\not{a} \times \not{a} \times a \times a \times a}{\not{a} \times \not{a}}$$
$$= a^3$$

i.e.
$$a^5 \div a^2 = a^{5-2}$$
$$= a^3$$

The general result is:

$$a^m \div a^n = a^{m-n}$$

Examples

Simplify:

(a) $a^{12} \div a^3$ (b) $a^4 \times a^5 \div a^7$ (c) $\frac{a^3b^6}{a^5b^2}$

Solutions (a) $a^{12} \div a^3 = a^9$

(b) $a^4 \times a^5 \div a^7 = \frac{a^4 \times a^5}{a^7} = \frac{a^9}{a^7} = a^2$

(c) Divide the numerator and the denominator by the lowest powers of a and b which appear in the expression. These are a^3 and b^2.

$$\frac{a^3b^6}{a^5b^2} = \frac{(a^3 \div a^3) \times (b^6 \div b^2)}{(a^5 \div a^3) \times (b^2 \div b^2)}$$
$$= \frac{1 \times b^4}{a^2 \times 1}$$
$$= \frac{b^4}{a^2}$$

You should now attempt questions 5 to 9 of exercise 3 on page 14.

LAW 3

Written out in full,

$$\left(a^2\right)^3 = a^2 \times a^2 \times a^2$$
$$= a^6$$

i.e. $\left(a^2\right)^3 = a^{2\times 3}$

The general result is

$$\left(a^m\right)^n = a^{mn}$$

Example

$$\left(a^7\right)^3 = a^{21}$$

LAW 4

Written out in full,

$$(ab)^3 = ab \times ab \times ab$$
$$= a^3b^3$$

The general result is

$$(ab)^m = a^m b^m$$

Examples

(a) $(2x)^3 = 2^3x^3 = 8x^3$

(b) $\left(3x^2y\right)^3 = 3^3\left(x^2\right)^3y^3$
$= 27x^6y^3$

You should now attempt question 10 to 14 of exercise 3 on page 14.

Multiplication of Directed Numbers.

If a directed number is multiplied on the right by a positive number, its sign is unchanged.

Thus $$4 \times 5 = 20$$
and $$(-4) \times 5 = -20$$

If a directed number is multiplied on the right by a positive number, its sign changes.

Thus $$4 \times (-5) = -20$$
and $$(-4) \times (-5) = 20$$

These results lead to the following rule:

The product of two numbers with like signs is positive, whilst the product of two numbers with unlike signs is negative.

Examples

(a) $7 \times (-4) = -28$

(b) $(-3)^2 = (-3) \times (-3) = 9$

(c) $$\begin{aligned}(-a) \times 3b &= (-1) \times a \times 3 \times b \\ &= (-1) \times 3 \times a \times b \\ &= -3ab\end{aligned}$$

(d) $$\begin{aligned}(-3p) \times 4q \times (-5r) &= -3 \times p \times 4 \times q \times (-5) \times r \\ &= -3 \times 4 \times (-5) \times pqr \\ &= -12 \times (-5) \times pqr \\ &= 60pqr\end{aligned}$$

You should now attempt questions 15 to 22 of exercise 3 on page 14.

Division of Directed Numbers

As $3 \times (-5) = -15$, it follows that

(i) $(-15) \div 3 = -5$ and
(ii) $(-15) \div (-5) = 3$

Results like these lead to the following rule:

When dividing, two numbers with like signs give a positive result whilst two numbers with unlike signs give a negative result.

Note that this rule is similar to the one for multiplication.

Examples

(a) $-9 \div 3 = -3$

(b) $\dfrac{9}{-3} = -3$

(c) $(-9) \div (-3) = 3$

(d) $\dfrac{-3}{4} = -\dfrac{3}{4}$

(e) $\dfrac{8}{-10} = -\dfrac{8}{10} = -\dfrac{4}{5}$

(f) $(-2a) \div 3b = \dfrac{-2a}{3b} = -\dfrac{2a}{3b}$

(g) $(3x) \div (-4y) = \dfrac{3x}{-4y} = -\dfrac{3x}{4y}$

(h) $(-p^4q^3) \div (-pq^5) = \dfrac{-p^4q^3}{-pq^5} = \dfrac{p^4q^3}{pq^5} = \dfrac{p^3}{q^2}$

(i) $\dfrac{4a \times (-3b)}{-2c} = \dfrac{-12ab}{-2c} = \dfrac{12ab}{2c} = \dfrac{6ab}{c}$

You should now attempt questions 23 to 34 of exercise 3 on page 14.

Exercise 3

Simplify:

(1) $a^4 \times a^6$ **(2)** $x^3 \times x^4 \times x^5$ **(3)** $3x^2y \times 4x^4y^3$ **(4)** $2xy^3 \times 3x^2y^2 \times 4x^3y$

Now turn to page 10

(5) $a^{12} \div a^3$ **(6)** $\frac{a^4 \times a^6}{a^3}$ **(7)** $\frac{a^2}{a^5}$ **(8)** $a^4b^3 \div a^2b^6$

(9) $6a^2b^4c^5 \div 9ab^6c^3$

Now turn to page 11

(10) $(x^4)^3$ **(11)** $(5x^3)^2$ **12)** $(2x^3y^2)^4$

(13) $(2xy^2)^3 \div (4x^2y)^2$ **(14)** $(2x^2y)^3 \times 3x^3y^2 \div (4xy^2)^2$

Now turn to page 12

Evaluate:

(15) $7 \times (-3)$ **(16)** $(-4) \times 6$ **(17)** $(-6) \times (-5)$ **(18)** $(-4)^2$ **(19)** $(-1)^3$

Simplify:

(20) $4bc \times (-ab)$ **(21)** $2x \times (-3y) \times 4z$ **(22)** $(-a^2b)^3$

Now turn to page 13

Evaluate, leaving your answers as a fraction in its lowest terms where appropriate:

(23) $16 \div (-2)$ **(24)** $(-14) \div 7$ **(25)** $(-18) \div (-3)$

(26) $\frac{-3 \times 4}{-2}$ **(27)** $\frac{(-4) \times (-6)}{2 \times (-3)}$ **(28)** $-7 \div 9$ **(29)** $\frac{9}{-12}$

Simplify:

(30) $-4x \div 6y$ **(31)** $\frac{2x}{-8y}$ **(32)** $(-6p^4y^2) \div 9py^7$

(33) $\frac{(-4a^2b) \times (-6a^4b^2)}{-16a^2b^4}$ **(34)** $\frac{(-2a^2bc) \times 5a^3b^2c^2}{4a^4c^3 \times (-3b^5)}$

Now start chapter 4 on page 15.

CHAPTER 4

REMOVING BRACKETS

Suppose you were asked to work out 55×102 without the aid of a calculator. The calculation could be carried out like this:

$$\begin{aligned} 55 \times 102 = 55 \times (100+2) &= 55 \times 100 + 55 \times 2 \\ &= 5500 + 110 \\ &= 5610 \end{aligned}$$

This method depends on the following law for removing brackets:

$$a(b+c) = ab + ac$$

i.e. multiply each term within the bracket by the quantity outside the bracket.

If the bracket is *followed* by the quantity outside, it can also be removed , so that :

$$(b+c)a = ba + ca$$

This technique is only valid if the terms within the bracket are separated by *plus* or *minus* signs: if the bracket in the expression $3 \times (y \times z)$ is removed, the result is simply $3yz$.

Examples

(a) $5(3x+4) = 15x + 20$

(b) $(3x-4y)x^2 = 3x^2 - 4yx^2$

(c) $p(2p-3q+4r) = 2p^2 - 3pq + 4pr$

(d) $$\begin{aligned}(3a^2 - 4ab + 5b^2)bc &= 3a^2 \times bc - 4ab \times bc + 5b^2 \times bc \\ &= 3a^2bc - 4ab^2c + 5b^3c\end{aligned}$$

You should now attempt questions 1 to 4 of exercise 4 on page 21.

Suppose we wished to remove the bracket in the expression $-(4p+5q-6r)$.
We can write:

$$\begin{aligned} -(4p+5q-6r) &= -1 \times (4p+5q-6r) \\ &= -1 \times 4p + (-1) \times 5q + (-1) \times (-6r) \\ &= -4p - 5q + 6r \end{aligned}$$

Notice that the signs of all the terms inside the bracket are changed when the bracket is removed. Results like these lead to the following rule:

When a bracketed expression has a minus sign in front of it the signs of all the terms inside the bracket are changed when the bracket is removed.

Note that this rule is only valid if the terms within the bracket are separated by *plus* or *minus* signs: the result of removing the bracket in the expression $-(p \times q)$ is simply $-pq$.

Examples

(a) $-(2p+3q)=-2p-3q$

(b) $-(x^2-3x)=-x^2+3x$

(c) $-3p(p^2-4q+2r)=-3p^3+12pq-6pr$

You should now attempt questions 5 to 8 of exercise 4 on page 21.

When simplifying expressions containing more than one bracket, remove the brackets and then add the like terms together:

Examples

(a) $3(2x+3y)-(x+5y)=6x+9y-x-5y$
$=5x+4y$

(b) $2x(x^2-3x)-3x(x^2-4)=2x^3-6x^2-3x^3+12x$
$=-x^3-6x^2+12x$

You should now attempt questions 9 to 12 of exercise 4 on page 21.

Expansion of two brackets

Expanding an expression means multiplying it out. For example, the result of expanding $x(x-2)$ is x^2-2x.

Suppose we had to expand the expression $(a+b)(c+d)$. Firstly, multiply each term in the second bracket by $(a+b)$. This gives:

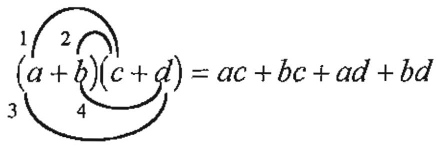
$$(a+b)(c+d)=(a+b)c+(a+b)d.$$

If we now remove the brackets on the right hand side we find that :

$$(a+b)(c+d)=ac+bc+ad+bd$$

Notice that each term in the first bracket is multiplied by each term in the second bracket. To make sure that you don't miss out any of the products appearing in such an expression, follow the same order every time.

The order used in the worked examples below is:



Examples Expand and simplify:

(a) $(3x+2)(x+4)$ (b) $(2y-3)(3y-2)$ (c) $(3p-4q)(2p+5q)$

Solutions To follow the order of appearance of the terms in the expansions, you may find it helpful to refer to the diagram above. The dots shown below represent multiplication signs and *not* decimal points.

(a) $$\begin{aligned}(3x+2)(x+4)&=3x.x+2.x+3x.4+2\,.\,4\\&=3x^2+2x+12x+8\\&=3x^2+14x+8\end{aligned}$$

(b) $$\begin{aligned}(2y-3)(3y-2)&=2y.3y-3\,.\,3y+2y(-2)+(-3)(-2)\\&=6y^2-9y-4y+6\\&=6y^2-13y+6\end{aligned}$$

(c) $$\begin{aligned}(3p-4q)(2p+5q)&=3p.2p-4q.2p+3p.5q-4q.5q\\&=6p^2-8pq+15pq-20q^2\\&=6p^2+7pq-20q^2\end{aligned}$$

With practice, you should develop the confidence to go straight to the simplified form.

You should now attempt questions 13 to 22 of exercise 4 on page 21.

Polynomial Expansions

Suppose we expand the expression $(3x^2+2)(4+x)$.

If the terms in the answer are written down in the order used in the previous section, we obtain:

$$(3x^2+2)(4+x)=12x^2+8+3x^3+2x$$

The result is an example of a *polynomial* in x. If the final answer to a problem is a polynomial, it is usual to rearrange the powers so that they are in descending order. The constant, if any, comes at the end of the expression.

It follows that the result of rearranging $12x^2+8+3x^3+2x$ is

$$3x^3+12x^2+2x+8$$

Examples

Expand and simplify:

(a) $(3-4x)(2-5x)$ (b) $(3-5x)(x+3)$ (c) $(2x^2+3)(1+x^3)$

Solutions In each case the final answer is given as a series of descending powers of x, followed by a constant.

(a) $$\begin{aligned}(3-4x)(2-5x)&=6-8x-15x+20x^2\\&=6-23x+20x^2\\&=20x^2-23x+6\end{aligned}$$

(b) $$\begin{aligned}(3-5x)(x+3)&=3x-5x^2+9-15x\\&=-5x^2-12x+9\end{aligned}$$

(c) $$\begin{aligned}(2x^2+3)(1+x^3)&=2x^2+3+2x^5+3x^3\\&=2x^5+3x^3+2x^2+3\end{aligned}$$

You should now attempt questions 23 to 26 of exercise 4 on page 21.

Three Important Expansions

An expression which is the sum or difference of two terms is called a *binomial*.

e.g. $2x+3y$

In algebra we often have to square binomials. Instead of multiplying out a pair of brackets every time, it is quicker to use a general result which we now derive.

If we expand $(a+b)^2$ we obtain

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + ba + ab + b^2$$

therefore:

$$(a+b)^2 = a^2 + 2ab + b^2$$

This result, and the others derived in this section, should be memorised.

They do <u>not</u> appear in the formula book issued for external examinations.

Examples

Expand (a) $(3x+4)^2$ (b) $(4x+3y)^2$

Solutions

(a) Put $a = 3x$ and $b = 4$ in the result above:

$$(3x+4)^2 = (3x)^2 + 2\,.\,3x\,.\,4 + 4^2$$
$$= 3^2x^2 + 24x + 16$$
$$= 9x^2 + 24x + 16$$

(b) $$(4x+3y)^2 = (4x)^2 + 2\,.\,4x\,.\,3y + (3y)^2$$
$$= 16x^2 + 24xy + 9y^2$$

You should now attempt questions 27 to 30 of exercise 4 on page 21.

If we expand $(a-b)^2$, we obtain:

$$(a-b)^2 = a^2 - 2ab + b^2$$

You should check this result by multiplying out $(a-b)(a-b)$

Example

Expand $(2-3x)^2$

Solution Put $a = 2$ and $b = 3x$ in the result above.

$$\begin{aligned}(2-3x) &= 2^2 - 2 . 2 . 3x + (3x)^2 \\ &= 4 - 12x + 9x^2 \\ &= 9x^2 - 12x + 4\end{aligned}$$

You should now attempt questions 31 to 34 of exercise 4 on page 21.

If we expand $(a+b)(a-b)$, we obtain $(a+b)(a-b) = a^2 + bc - ab - b^2$ which gives:

$$(a+b)(a-b) = a^2 - b^2$$

*This result is called "**The difference of two squares**".*

Example

Expand (a) $(2x+3)(2x-3)$ (b) $(4-3x^2)(4+3x^2)$

Solutions (a) Put $a = 2x$ and $b = 3$ in the result above.

$$\begin{aligned}(2x+3)(2x-3) &= (2x)^2 - 3^2 \\ &= 4x^2 - 9\end{aligned}$$

(b) $$\begin{aligned}(4-3x^2)(4+3x^2) &= 4^2 - (3x^2)^2 \\ &= 16 - 3^2(x^2)^2 \\ &= 16 - 9x^4\end{aligned}$$

You should now attempt questions 35 to 40 of exercise 4 on page 21.

Exercise 4

Remove the brackets in the following expressions:

(1) $7(8x-4)$ **(2)** $(4a-3b)c$ **(3)** $x^3(2x^2-3x+1)$ **(4)** $(2x^2-3xy+4y^2)xy$

Now return to page 15.

Remove the brackets and simplify:

(5) $-(3a+4b)$ **(6)** $-(4y^2-3)$ **(7)** $-7p(2p^2-p+1)$ **(8)** $-3m^2(3m-2n-1)$

Now return to page to page 16.

Remove the brackets in the following expressions:

(9) $-3(4a-2b)+4(a-b)$ **(10)** $4(1-2x)-3(2x-1)$

(11) $3x(x-4)-4x(x-5)$ **(12)** $3x(x^2-5x+1)-2x(x^2+3)-3(x^2+6)$

Now return to page 17.

Expand and simplify

(13) $(x+2)(x+3)$ **(14)** $(y-3)(y-4)$ **(15)** $(p-4)(p+1)$

(16) $(r+8)(r-3)$ **(17)** $(4x+3)(3x+2)$ **(18)** $(3y-4)(2y-5)$

(19) $(2x-5)(3x+4)$ **(20)** $(3x-4y)(2x-3y)$

(21) $(a-4b)(a+5b)$ **(22)** $(2a-3b)(2a+3b)$

Now return to page 18.

Expand and simplify:

(23) $(2-3x)(4-5x)$ **(24)** $(2-3x)(3+2x)$ **(25)** $(3x^2+1)(x-6)$

(26) $(4x^3-5)(2-3x)$

Now return to page 19.

Expand:

(27) $(4x+1)^2$ **(28)** $(3x+5)^2$ **(29)** $(2p+3q)^2$ **(30)** $(3x^2+2)^2$

Now return to page 20.

Expand:

(31) $(3x-1)^2$ **(32)** $(3-4x)^2$ **(33)** $(2x-3y)^2$ **(34)** $(4x^3-1)^2$

Now return to page 20.

Expand:

(35) $(4x-1)(4x+1)$ **(36)** $(3x-4)(3x+4)$ **(37)** $(2+5x)(2-5x)$

(38) $(4p-3q)(4p+3q)$ **(39)** $(2x^3-3)(2x^3+3)$ **(40)** $(3p^2+2q^2)(3p^2-2q^2)$

Now turn to chapter 5 on page 22.

CHAPTER 5

SOLVING LINEAR EQUATIONS

Consider the equation $7x - 5 = 4x + 3$. The terms are either multiples of x or constants. Such an equation is said to be *linear in* x.

When solving equations, remember that

(1) we can add the same quantity to both sides of the equation.
(2) we can subtract the same quantity from both sides of the equation.
(3) we can multiply or divide both sides of the equation by the same quantity.

In each case, both sides of the equation remain equal.

Examples Solve the following equations:

(a) $x - 3 = 9$ (b) $\frac{z}{5} = 12$ (c) $3m + 5 = 17$

(d) $2 - \frac{3}{4}x = 7$ (e) $4x - 1 = 3 - 7x$

Solutions

(a)
$$x - 3 = 9$$
$$\therefore x - 3 + \mathbf{3} = 9 + \mathbf{3} \quad \text{(adding 3 to both sides)}$$
$$\therefore \underline{x = 12}$$

(b)
$$\frac{z}{5} = 12$$
$$\therefore \mathbf{5} \times \frac{z}{5} = \mathbf{5} \times 12 \quad \text{(multiplying both sides by 5)}$$
$$\therefore \underline{z = 60}$$

(c)
$$3m + 5 = 17$$
$$\therefore 3m = 12 \quad \text{(subtracting 5 from both sides)}$$
$$\therefore \underline{m = 4} \quad \text{(dividing both sides by 3)}$$

(d)
$$2 - \frac{3}{4}x = 7$$
$$\therefore -\frac{3}{4}x = 5 \quad \text{(subtracting 2 from both sides)}$$
$$\therefore -3x = 20 \quad \text{(multiplying both sides by 4)}$$
$$\therefore x = \frac{20}{-3} \quad \text{(dividing both sides by } -3\text{)}$$
$$\therefore \underline{x = -6\frac{2}{3}}$$

Note that the final answer is left as a mixed number. If we attempt to write it as a decimal, the result will only be approximate.

(e) $4x-1=3-7x$

$\therefore 11x-1=3$ (adding $7x$ to both sides)

$\therefore 11x=4$ (adding 1 to both sides)

$\therefore x=\dfrac{4}{11}$ (dividing both sides by 11)

You should now attempt questions 1 to 11 of exercise 5 on page 25.

If an equation contains brackets, remove them first.

Example Solve: $3(2x+4)-5(3x-2)=7$

Solution

$$6x+12-15x+10=7$$
$$\therefore -9x+22=7$$
$$\therefore -9x=-15$$
$$\therefore x=\frac{-15}{-9}$$
$$\therefore x=\frac{5}{3}$$
$$\therefore x=1\frac{2}{3}$$

You should now attempt questions 12 to 14 of exercise 5 on page 25.

If an equation contains fractions, remove them by multiplying each term by a multiple of all the denominators. To make the algebra as simple as possible, you should find the *lowest common multiple* (LCM).

Examples Solve the equations:

(a) $\dfrac{x}{4}+\dfrac{3}{5}=\dfrac{3x}{2}-1$ (b) $\dfrac{m-4}{3}-\dfrac{2m-1}{2}=5$ (c) $\dfrac{2}{2x+1}=\dfrac{3}{5}$

Solutions (a) The lowest common multiple (LCM) of 4,5 and 2 is 20.

Multiplying each term by 20,

$$\frac{\mathbf{20}\times x}{4}+\frac{\mathbf{20}\times 3}{5}=\frac{\mathbf{20}\times 3x}{2}-\mathbf{20}\times 1$$

Dividing **20** by each of the denominators gives:

$$5x+4\times 3=10\times 3x-20$$
$$\therefore 5x+12=30x-20$$
$$\therefore 32=25x$$
$$\therefore x=\frac{32}{25}=1\frac{7}{25}$$

(b) $$\frac{m-4}{3}-\frac{2m-1}{2}=5$$

The LCM of the denominator is 6. Multiplying each term by 6,

$$\frac{6(m-4)}{3}-\frac{6(2m-1)}{2}=6\times 5$$

$$2(m-4)-3(2m-1)=30$$

$$2m-8-6m+3=30$$

$$-4m-5=30$$

$$-4m=35$$

$$m=\frac{35}{-4}$$

$$m=-8\frac{3}{4}$$

(c) $$\frac{2}{2x+1}=\frac{3}{5}$$

Multiply each term by $5(2x+1)$

$$\frac{2\times 5(\mathbf{2x+1})}{2x+1}=\frac{3\times 5(\mathbf{2x+1})}{5}$$

$$\therefore\ 2\times 5=3\times(2x+1)$$

$$\therefore\ 10=6x+3$$

$$\therefore\ 7=6x$$

$$\therefore\ x=\frac{7}{6}=1\frac{1}{6}$$

You should now attempt questions 15 to 22 of exercise 5 on page 25.

Exercise 5

Solutions which are not integers should be given as either fractions or mixed numbers.

Solve the following equations:

(1) $x-2=6$ **(2)** $z+4=1$ **(3)** $\frac{m}{3}=4$ **(4)** $6p=-18$

(5) $\frac{3x}{4}=5$ **(6)** $3y+4=13$ **(7)** $7x+13=2$ **(8)** $13-3x=25$

(9) $\frac{2}{3}-9=-13$ **(10)** $5-4p=2p-7$ **(11)** $8z+2=5z-14$

Return to page 23

Solve the following equations:

(12) $2(3x+5)=11$ **(13)** $3(y-1)-4(2y+3)=14$ **(14)** $4(x-5)=7-(3-2x)$

Return to page 23.

Solve the following equations:

(15) $\frac{m}{5}-\frac{m}{3}=2$ **(16)** $\frac{z}{2}+\frac{z}{3}+1=\frac{z}{6}$ **(17)** $\frac{y+1}{3}+\frac{2y-1}{4}=2$

(18) $\frac{3(2z-1)}{4}-\frac{2(z+2)}{3}=1$ **(19)** $\frac{4p}{15}-\frac{p-6}{6}=3+\frac{3p}{10}$

(20) $\frac{3}{2x-1}=\frac{1}{4}$ **(21)** $\frac{3x-1}{4x+3}=\frac{19}{34}$ **(22)** $\frac{2}{m}+\frac{5}{2m}+\frac{4}{3m}=10$

ANSWERS

Exercise 1

1.	$12x$	**2.**	$21y$	**3.**	$12pq$	**4.**	$63ab$
5.	$6xyz$	**6.**	$60pqr$	**7.**	$6a^2$	**8.**	$12b^5$
9.	$12x^2yz$	**10.**	$10x^5y^3$	**11.**	$16a^2$	**12.**	$9a^2b^2$
13.	$144p^4q^2$	**14.**	$9x^4$	**15.**	$8a^6b^3$		
16.	$2x$	**17.**	$2xy$	**18.**	$4x^2$	**19.**	$5ab$
20.	$\frac{2}{3}y$	**21.**	$\frac{3}{4}ab$	**22.**	$\frac{6xy}{7}$	**23.**	$\frac{2ab^2}{3}$
24.	$\frac{2x}{3}$	**25.**	$\frac{2}{3y}$	**26.**	$\frac{2a}{3b}$	**27.**	$\frac{3q}{4}$
28.	$6x$	**29.**	$\frac{4}{5b}$	**30.**	$\frac{2a^2}{3b^3}$	**31.**	$\frac{4a^2}{5b^2c}$
32.	$\frac{p^3}{q}$						

Exercise 2

1. $18x$	**2.** $12ab$	**3.** $2ab^2$	**4.** $2x^2+8x+2$
5. $4p^2q+2pqr$	**6.** $6a^2+5ab+2b^2$	**7.** -4	**8.** -12
9. -5	**10.** $-2x^2$	**11.** $3a^2b+2ab^2$	**12.** $-7a^2+5ab+4b^2$
13. -7	**14.** -5	**15.** 11	**16.** -15
17. -7	**18.** $-3x^2-x$	**19.** $-9a^2-3ab+b^2$	**20.** $-4a^2b-8ab^2-2b^3$

Exercise 3

1. a^{10}	**2.** x^{12}	**3.** $12x^6y^4$	**4.** $24x^6y^6$	**5.** a^9	**6.** a^7
7. $\frac{1}{a^3}$	**8.** $\frac{a^2}{b^3}$	**9.** $\frac{2ac^2}{3b^2}$	**10.** x^{12}	**11.** $25x^6$	**12.** $16x^{12}y^8$
13. $\frac{y^4}{2x}$	**14.** $\frac{3}{2}x^7y$	**15.** -21	**16.** -24	**17.** 30	**18.** 16
19. -1	**20.** $-4ab^2c$	**21.** $-24xyz$	**22.** $-a^6b^3$	**23.** -8	**24.** -2
25. 6	**26.** 6	**27.** -4	**28.** $-\frac{7}{9}$	**29.** $-\frac{3}{4}$	**30.** $-\frac{2x}{3y}$
31. $-\frac{x}{4y}$	**32.** $-\frac{2p^3}{3y^5}$	**33.** $-\frac{3a^4}{2b}$	**34.** $\frac{5a}{6b^2}$		

Exercise 4

1. $56x-28$ **2.** $4ac-3bc$ **3.** $2x^5-3x^4+x^3$

4. $2x^3y-3x^2y^2+4xy^3$ **5.** $-3a-4b$ **6.** $-4y^2+3$

7. $-14p^3+7p^2-7p$ **8.** $-9m^3+6m^2n+3m^2$ **9.** $-8a+2b$

10. $7-14x$ **11.** $-x^2+8x$ **12.** $x^3-18x^2-3x-18$

13. x^2+5x+6 **14.** $y^2-7y+12$ **15.** p^2-3p-4

16. $r^2+5r-24$ **17.** $12x^2+17x+6$ **18.** $6y^2-23y+20$

19. $6x^2-7x-20$ **20.** $6x^2-17xy+12y^2$ **21.** $a^2+\not{0}ab-20b^2$

22. $4a^2-9b^2$ **23.** $15x^2-22x+8$ **24.** $-6x^2-5x+6$

25. $3x^3-18x^2+x-6$ **26.** $-12x^4+8x^3+15x-10$ **27.** $16x^2+8x+1$

28. $9x^2+30x+25$ **29.** $4p^2+12pq+9q^2$ **30.** $9x^4+12x^2+4$

31. $9x^2-6x+1$ **32.** $16x^2-24x+9$ **33.** $4x^2-12xy+9y^2$

34. $16x^6-8x^3+1$ **35.** $16x^2-1$ **36.** $9x^2-16$

37. $4-25x^2$ **38.** $16p^2-9q^2$ **39.** $4x^6-9$

40. $9p^4-4q^4$

Exercise 5

1 $x=8$ **2** $z=-3$ **3** $m=12$ **4** $p=-3$ **5** $x=6\frac{2}{3}$ **6** $y=3$

7 $x=-1\frac{4}{7}$ **8** $x=-4$ **9** $x=-6$ **10** $p=2$ **11** $z=-5\frac{1}{3}$ **12** $x=\frac{1}{6}$

13 $y=-5\frac{4}{5}$ **14** $x=\overset{12}{\not{-2}}$ **15** $m=-15$ **16** $z=-1\frac{1}{2}$ **17** $y=2\frac{3}{10}$ **18** $z=3\frac{7}{10}$

19 $p=-10$ **20** $x=6\frac{1}{2}$ **21** $x=3\frac{1}{2}$ **22** $m=\frac{7}{12}$

CHAPTER 6

TRANSPOSITION OF FORMULAE

A formula is an equation which relates two or more variables. For example, $A = lb$ expresses the area (A) of a rectangle in terms of its length (l) and breadth (b).

A is called the *subject* of the formula. If the values of l and b are given, they can be substituted in the formula to find the value of A.

Suppose the values of A and l are given and we require the corresponding value of b. It is often convenient to rearrange the formula so that b becomes the subject. This process is called *transposition*.

If you need to change the subject, think of the formula as an equation in which the required subject is the unknown. You can then apply the methods which you practised in the previous chapter.

Examples

(a) Transpose $V = IR$ for I

Solution: $I = \dfrac{V}{R}$ (Dividing both sides by R)

(b) Transpose $A = 2\pi rh$ for r

Solution: $A = 2\pi hr$ (Rearranging right hand side)

$\therefore\ r = \dfrac{A}{2\pi h}$ (Dividing both sides by $2\pi h$)

(c) Transpose $P = \dfrac{W}{nt}$ for t

Solution: $Pnt = W$ (Multiplying both sides by nt)

$\therefore\ t = \dfrac{W}{Pn}$ (Dividing both sides by Pn)

(d) Transpose $V = \pi r^2 h + abc$ for a

Solution: $V - \pi r^2 h = abc$ (Subtracting $\pi r^2 h$ from both sides)

$\therefore\ a = \dfrac{V - \pi r^2 h}{bc}$ (Dividing both sides by bc)

You should now attempt questions 1 to 9 of exercise 6 on page 30.

Example

Transpose $v^2 = u^2 - 2rs$ for s

Solution:

$$v^2 - u^2 = -2rs \qquad \text{(subtracting } u^2 \text{ from both sides)}$$

$$\therefore \quad s = \frac{v^2 - u^2}{-2r} \qquad \text{(dividing both sides by } -2r\text{)}$$

$$= \frac{-(v^2 - u^2)}{2r} \qquad \text{(multiplying numerator and denominator by } -1\text{)}$$

$$= \frac{u^2 - v^2}{2r} \qquad \text{(removing bracket and rearranging)}$$

The solution became quite complicated because of the minus sign originally appearing before the term containing the required subject. The algebra is easier if our first step is to *add* this term to both sides.

Thus $v^2 = u^2 - 2rs$

$$\therefore \quad v^2 + 2rs = u^2 \qquad \text{(adding } 2rs \text{ to both sides)}$$

$$\therefore \quad 2rs = u^2 - v^2 \qquad \text{(subtracting } v^2 \text{ from both sides)}$$

$$\therefore \quad s = \frac{u^2 - v^2}{2r} \qquad \text{(dividing both sides by } 2r\text{)}$$

You should now attempt questions 10 to 12 of exercise 6 on page 30.

Examples involving brackets

(a) Transpose $F = \frac{9}{5}C + 32$ for C

Solution:

$$F - 32 = \frac{9}{5}C \qquad \text{(subracting 32 from both sides)}$$

$$\therefore \quad 5(F - 32) = 9C \qquad \text{(multiplying both sides by 5)}$$

$$\therefore \quad C = \frac{5}{9}(F - 32) \qquad \text{(dividing both sides by 9)}$$

The final answer can be left in this form: there is no need to remove the bracket in the denominator.

Example

(b) Transpose $M = P\left(1+\frac{r}{100}\right)$ for r

Solution: If the required subject appears within a bracket a neat expression for it can be obtained for it by removing that bracket. Thus:

$$M = P + \frac{\Pr}{100}$$

$$\therefore \quad M - P = \frac{\Pr}{100} \text{(subtracting } P \text{ from both sides)}$$

$$\therefore \quad \Pr = 100(M-P) \quad \text{(multiplying both sides by100)}$$

$$\therefore \quad r = 100\frac{(M-P)}{P} \quad \text{(dividing both sides by } P\text{)}$$

The final answer can be left in this form. If you wish , you can now divide each term in the bracket by P to obtain:

$$r = 100\left(\frac{M}{P} - \frac{P}{P}\right)$$

$$= 100\left(\frac{M}{P} - 1\right)$$

You should now attempt questions 13 to 17 of exercise 6 On page 30.

Examples involving Square Roots

(a) Transpose $w^2 = u^2 - v^2$ for u

Solution: $w^2 + v^2 = u^2$ (adding v^2 to both sides)

$u = \sqrt{w^2 + v^2}$ (taking the square root of both sides)

Note: Some students now try to simplify $\sqrt{w^2 + v^2}$ and obtain $w + v$.

THIS IS NONSENSE!

eg. If we put $w = 3$ and $v = 4$, then

$$\sqrt{w^2 + v^2} = \sqrt{9+16} = \sqrt{25} = 5$$

but $w + v = 3 + 4 = 7$

$\therefore \sqrt{w^2 + v^2}$ *DOES NOT EQUAL* $w + v$

Examples

(b) Transpose $a^2 = 3b^2 - 4c^2$ for c

Solution:

$$4c^2 + a^2 = 3b^2 \quad \text{(adding } 4c^2 \text{ to both sides)}$$

$$\therefore \quad 4c^2 = 3b^2 - a^2 \quad \text{(subtracting } a^2 \text{ from both sides)}$$

$$\therefore \quad c^2 = \frac{3b^2 - a^2}{4} \quad \text{(dividing both sides by 4)}$$

$$\therefore \quad c = \sqrt{\frac{3b^2 - a^2}{4}} \quad \text{(taking the square root of both sides)}$$

As the denominator is a perfect square, we can simplify this result by taking the square root of the numerator and denominator separately.

Thus

$$c = \frac{\sqrt{3b^2 - a^2}}{\sqrt{4}}$$

$$= \frac{\sqrt{3b^2 - a^2}}{2}$$

(c) Transpose $d = \sqrt{2ab}$ for a

Solution: Remember that when the square root of an expression is squared, all that happens is that the root sign disappears.

Thus

$$d^2 = 2ab \quad \text{(squaring both sides)}$$

$$\therefore \quad a = \frac{d^2}{2b} \quad \text{(dividing both sides by } 2b\text{)}$$

(d) Transpose $a = b + 3\sqrt{\frac{c}{d}}$ for d

Solution:

$$a - b = 3\sqrt{\frac{c}{d}} \quad \text{(subtracting } b \text{ from both sides)}$$

$$\therefore \quad (a-b)^2 = \left(3\sqrt{\frac{c}{d}}\right)^2 \quad \text{(squaring both sides)}$$

$$= 3^2\left(\sqrt{\frac{c}{d}}\right)^2 \quad \text{(remember that } (pq)^2 = p^2q^2\text{)}$$

$$= 9\frac{c}{d}$$

$$\therefore \quad d(a-b)^2 = 9c \quad \text{(multiplying both sides by } d\text{)}$$

$$\therefore \quad d = \frac{9c}{(a-b)^2} \quad \text{(dividing both sides by } (a-b)^2\text{)}$$

You should now attempt questions 18 to 24 of exercise 6 on page 30.

Exercise 6

In each of the following questions transpose the formula to make the subject the letter shown in brackets.

(1) $F = ma \quad (a)$ **(2)** $A = 4ab \quad (b)$ **(3)** $V = \pi r^2 h \quad$ (h)

(4) $y = \frac{x}{b} \quad (x)$ **(5)** $P = \frac{RT}{V} \quad (T)$ **(6)** $i = \frac{q}{t} \quad (t)$

(7) $z = \frac{ab}{cd} \quad (d)$ **(8)** $v = u + at \quad (a)$ **(9)** $v^2 = u^2 + 2ax \quad (x)$

Now turn to page 27.

Transpose the following formulae for the letters shown in brackets:

(10) $w = v - pg \quad (g)$ **(11)** $p = \frac{mv - mu}{t} \quad (u)$

(12) $a^2 = b^2 + c^2 - 2bc\cos A \quad (\cos A)$

Now turn to page 28.

Transpose the following formulae for the letters shown in brackets:

(13) $y = m + \frac{x}{n} \quad (x)$ **(14)** $x = ut + \frac{1}{2}at^2 \quad (a)$

(15) $A = \pi r(r + h) \quad (h)$ **(16)** $T = \frac{6(D - d)}{l} \quad (d)$

(17) $w = v(1 + at) \quad (t)$

Now turn to page 29.

Transpose the following formulae for the letters shown in brackets:

(18) $4p^2 = 3q^2 + 16r^2 \quad (r)$ **(19)** $T = F + mrw^2 \quad (w)$

(20) $v = \sqrt{\frac{6h}{a}} \quad (h)$ **(21)** $T = 2\pi\sqrt{\frac{l}{g}} \quad (l)$

(22) $T = 2\pi\sqrt{\frac{I}{Mgh}} \quad (g)$ **(23)** $m = n + t\sqrt{u} \quad (u)$

(24) $z = \sqrt{\frac{w}{x + y}} \quad (y)$

Now turn to chapter 7 on page 31.

CHAPTER 7

INTRODUCTION TO FACTORISATION

The number 24 can be written as a product in many different ways. For example,

$$24 = 4 \times 6$$

The numbers 4 and 6 are said to be *factors* of 24.

It is often convenient to think of an algebraic expression as a product of factors. For example, $6a^2b$ can be regarded as the product of the factors 6, a^2 and b.

Consider the expression $4x + 4y$.
The number 4 is a *common factor* of both terms.
It is also a factor of the complete expression as $4x + 4y = 4(x + y)$.
The other factor is $(x + y)$.
The process of expressing an algebraic sum as a product of factors is called *factorisation.*

Examples Factorise:

(a) $3x + 12$ (b) $ax - ay$ (c) $3x^2 + 5x$

Solutions (a) 3 is a factor of both terms and is therefore a factor of the complete expression. The other factor is contained within a bracket.

To find the terms inside the bracket, divide each of the terms in $3x + 12$ by 3. Thus

$$3x + 12 = 3(x + 4)$$

(b) $ax - ay = a(x - y)$

(c) x is a common factor of $3x^2$ and $5x$. Thus

$$3x^2 + 5x = x(3x + 5)$$

You should now attempt questions 1 to 6 on page 34.

Highest Common Factor (HCF)

Suppose we wished to factorise $6x^2 - 24x$. Now 6 and x are both common factors of $6x^2$ and $24x$. Their product $6x$ is therefore a common factor of both terms and can be placed outside the bracket containing the other factor. Thus

$$6x^2 - 24x = 6x(x - 4)$$

$6x$ is the *highest common factor (HCF)* of $6x^2$ and $24x$.

When factorising an algebraic expression, you should find the highest common factors of all the terms.

Examples Find the HCF of:

(a) $xy^2z^3,\quad x^2y^3z^2,\quad x^3y^4z^2$

(b) $6x^2y,\quad 9x^3,\quad 12x^4y^2$

Solutions (a) Each term contains powers of x, y and z. To find the HCF choose the *lowest* power of each variable which occurs in the three terms and multiply them together. The lowest power of x is x, the lowest power of y is y^2 and the lowest power of z is z^2.
The HCF is therefore xy^2z^2.

(b) The highest common factor of the numbers 6,9 and 12 is 3.
The lowest power of x appearing in all three terms is x^2.
The variable y does not appear in the HCF as it only appears in two of the terms.
The HCF is therefore $3x^2$.

You should now attempt questions 7 to 10 on page 34.

Examples Factorise: (a) $21x^3y^2 - 14x$ (b) $11x^3y - 22xy^3 - 11xyz^2$

Solutions (a) The HCF of $21x^3y^2$ and $14x$ is $7x$.
Therefore $7x$ is a factor of $21x^3y^2 - 14x$.
To find the terms in the second factor, divide $21x^3y^2$ and $14x$ by $7x$.
Thus $21x^3y^2 - 14x = 7x(3x^2y^2 - 2)$

(b) The HCF of $11x^3y$, $22xy^3$ and $11xyz^2$ is $11xy$.
$\therefore\quad 11x^3y - 22xy^3 - 11xyz^2 = 11xy(x^2 - 2y^2 - z^2)$

You should now attempt questions 11 to 15 on page 34.

You may need to transpose formulae in which the required subject appears more than once. Some examples of this type can be solved by factorising.

Example Transpose $A = \dfrac{3B + 4C}{B + 5}$ for B

Solution

$A(B+5) = 3B + 4C$ (mutiplying both sides by $B + 5$)

$\therefore \quad AB + 5A = 3B + 4C$ (mutiplying out the bracket)

$\therefore \quad AB - 3B + 5A = 4C$ (subtracting $3B$ from both sides)

$\therefore \quad AB - 3B = 4C - 5A$ (subtracting 5A from both sides)

$\therefore \quad (A - 3)B = 4C - 5A$ (factorising left hand side)

$\therefore \quad B = \dfrac{4C - 5A}{A - 3}$ (dividing both sides by $A - 3$)

You should now attempt questions 16 to 20 on page 34.

Exercise 7

Factorise the following:

(1)	$5x+25$	**(2)**	$12x-16y$	**(3)**	$pa+pb$
(4)	$9a+12b-15c$	**(5)**	$5x^2-12x$	**(6)**	$6y^3+7y$

Now turn to page 32.

Find the HCF of the following sets of terms:

(7)	$5x^2, 10x$	**(8)**	$8x^3, 12x^4$
(9)	$a^3b^2c^3, a^2bc^2, a^4b^3c^4$	**(10)**	$4x^2y^2, 8x^3, 12x^4y^3$

Now return to page 32.

Factorise the following:

(11)	$18x^2-27x$	**(12)**	$4pq^3-6pq^2$
(13)	$3x^3-6x^2+12x$	**(14)**	$24ab^2+8ac^2-4ad^2$
(15)	$6ab^3-9abc^2+3ab^2c$		

Now turn to page 33.

Transpose the following formulae for the letters shown in brackets:

(16) $A=\dfrac{3P+5Q}{2P-1}$ (P)

(17) $F+mg=\dfrac{mv^2}{r}$ (m)

(18) $y=\dfrac{dx^2}{z-x^2}$ (x) Assume x is positive.

(19) $z=\sqrt{\dfrac{x}{x+y}}$ (x)

(20) $\dfrac{M}{m}=\sqrt{\dfrac{g+p}{g-p}}$ (g)

Now turn to chapter 8 on page 35.

CHAPTER 8

FACTORISATION OF QUADRATICS

Consider the expression $3x^2 - 8x + 5$. This is an example of a *quadratic in* x.
The *coefficient of* x^2 in the expression is the number 3 multiplying x^2. Similarly, the coefficient of x in the quadratic is -8. The *constant term* is the number 5 at the end.
In pure mathematics it is often necessary to factorise quadratic expressions. We shall begin with quadratics in which the coefficient of x^2 is 1.

Case 1 You should verify that $(x+2)(x+3) = x^2 + 5x + 6$.

Notice that : (i) the *product* of the numbers in the brackets is the *constant* term on the right hand side.
(ii) the *sum* of the numbers in the brackets is the *coefficient of* x.

Example Factorise $x^2 + 7x + 12$

Solution (i) Write the expression as the product of two brackets, leaving spaces for numbers to be inserted .
i.e. $x^2 + 7x + 12 = (x + \quad)(x + \quad)$
(ii) Find two numbers whose product is 12 and whose sum is 7. These are 4 and 3.
(iii) Insert these numbers into the brackets.
Thus $x^2 + 7x + 12 = (x+4)(x+3)$

Note: When factorising quadratics, check that your solution is correct by mentally multiplying out the brackets.

You should now attempt questions 1 to 4 of exercise on page 41

Case 2 You should verify that $(x-4)(x-3) = x^2 - 7x + 12$.

When the coefficient of x in a quadratic is *negative* and the constant term is *positive,* both brackets of the factorised form contain *minus* signs.
In the above expansion,the *product* of the numbers 4 and 3 within the brackets is the constant term; their *sum* is the 7 in the middle term.

Example Factorise $p^2 - 10p + 21$

Solution (i) Write $p^2 - 10p + 21 = (p - \quad)(p - \quad)$
(ii) Find two numbers whose product is 21 and whose sum is 10. These are 7 and 3
(iii) Insert these numbers into the brackets.
Thus $p^2 - 10p + 21 = (p-7)(p-3)$

You should now attempt questions 5 to 8 of exercise 8 on page 41

Case 3 You should verify that:

(i) $(x+6)(x-4)=x^2+2x-24$ and
(ii) $(x+3)(x-7)=x^2-4x-21$

In both cases, the constant term on the right hand side is negative. When this happens the signs in the brackets of the factorised form are different.
In example (i), the difference of the numbers 6 and 4 within the brackets is the 2 in the middle term.
In example (ii), the difference of the numbers 3 and 7 within the brackets is the 4 in the middle term.

Example $x^2+4x-21$

Solution

(i) Write $x^2+4x-21=(x+\quad)(x-\quad)$

(ii) Find two numbers whose product is 21 and whose difference is 4. These are 7 and 3

(iii) Insert these numbers into the brackets. As the coefficient of x in the quadratic is positive, the larger number, 7, should be put in the bracket containing the positive sign.
Thus $x^2+4x-21=(x+7)(x-3)$

Example Factorise $x^2-8x-20$

Solution

(i) Write $x^2-8x-20=(x+\quad)(x-\quad)$

(ii) Find two numbers whose product is 20 and whose difference is 8. These are 10 and 2.

(iii) As the coefficient of x in the quadratic is negative, insert the larger number, 10, into the bracket containing the minus sign. Put the 2 in the other bracket.
Thus $x^2-8x-20=(x+2)(x-10)$

You should now attempt questions 9 to 16 of exercise 8 on page 41

More difficult factorisation

You should verify that: $(3x+2)(x+4) = 3x^2 + 14x + 8$

The working can be displayed using the following table:

	coefficient of x	constant term
1st bracket	3	2
2nd bracket	1	4

The coefficient of x^2 is the product of the numbers 3 and 1 in the first column.
The constant term in the quadratic is the product of the numbers 2 and 4 in the second column.

The coefficient of x is the sum of the 'cross products' indicated by the lines i.e.

$$14 = 3 \times 4 + 1 \times 2$$

We now use an extension of this table to factorise quadratics in which the coefficient of x^2 is a prime number.

Example Factorise $5x^2 + 16x + 12$

Solution (a) Construct the following table:

5	1	12	2	6	3	4
1	12	1	6	2	4	3

The column on the left of the line consists of the pair of factors 5 and 1 of the coefficient of x^2. These need only be written down in the *one* particular order.
To the right of the line, all the possible pairs of factors of the constant term 12 are written down systematically. Notice that *both* orders for each pair are shown.

(b) For each column of numbers to the right of the line, calculate mentally the sum of the cross products with the factors to the left of the line. Stop when you obtain the coefficient of x, i.e. 16.

Working from left to right, the sums are:

$$5 \times 12 + 1 \times 1 = 61$$

$$5 \times 1 + 1 \times 12 = 17$$

$$5 \times 6 + 1 \times 2 = 32$$

$$5 \times 2 + 1 \times 6 = 16$$

(c) Highlight the columns producing the sum of 16 as shown below:

(5)	1	12	2	(6)	3	4
(1)	12	1	6	(2)	4	3

The highlighted numbers on the first row represent the factor $5x+6$ of the original quadratic. Those on the second row represent the factor $x+2$.

Thus $\quad 5x^2+16x+12=(5x+6)(x+2)$

When factorising quadratics it is often unnecessary to write down all the possible pairs of factors of the constant term. In the next example, you will be shown a quicker version of the method.

Example Factorise $3x^2+11x+8$

Solution
(i) Write down the pair of factors of 3 in one particular order.
(ii) Write down systematically pairs of factors of the constant term 8, *calculating the sums of the cross products as you go along.* Stop when you obtain the coefficient of x.
The working is shown below.

3	|	1	8
1	|	8	1

Thus $\quad 3x^2+11x+8=(3x+8)(x+1)$

You should now attempt question 17 to 20 of exercise 8 on page 41

Example Factorise $3x^2-16x+20$

Solution As the constant term is positive, we still calculate *sums* of cross products, stopping when we obtain 16. The signs in both brackets of the factorised form are negative.

3	|	1	20	2	10
1	|	20	1	10	2

Thus $3x^2-16x+20=(3x-10)(x-2)$

You should now attempt questions 21 to 24 of exercise 8 on page 41

Example Factorise $3x^2 - 8x - 28$

Solution If the constant term is *negative*, we work out the *differences* of the cross products. The signs in the brackets of the factorised form will be different.

(3)	|	1	28	2	(14)
(1)	|	28	1	14	(2)

$[1 \times 14 - 3 \times 2 = 8]$

Find the correct sign for each bracket by trial and error.
$(3x + 14)(x - 2)$ gives 8 for the coefficient of x when expanded but
$(3x - 14)(x + 2)$ gives -8 for the coefficient of x.

Thus $3x^2 - 8x - 28 = (3x - 14)(x + 2)$

You should now attempt questions 15 to 28 of exercise 8 on page 41

If the coefficient of x^2 is not a prime number, you may have to try more than one pair of its factors.

Example Factorise $12x^2 - 23x + 10$

Solution

(3)	2	1	|	1	10	(2)	5
(4)	6	12	|	10	1	(5)	2

Pairs of factors of 12 are written down systematically until 23 is obtained for the sum of the cross products. It isn't necessary to reverse the order of these factors.

Thus $12x^2 - 23x + 10 = (3x - 2)(4x - 5)$

You should now attempt questions 29 to 32 of exercise 8 on page 41

Examples Factorise the following quadratics:

(a) $3x^2 - 6x - 9$ (b) $10 + 3x - x^2$

(a) As 3 is a common factor of each term, factorise in two stages.

$$\begin{aligned} 3x^2 - 6x - 9 &= 3(x^2 - 2x - 3) \\ &= 3(x - 3)(x + 1) \end{aligned}$$

(b) As the coefficient of x^2 is negative, take out a factor of -1 from the complete expression and change the signs of all the terms of the quadratic.

$$\begin{aligned} 10 + 3x - x^2 &= -(x^2 - 3x - 10) \\ &= -(x-5)(x+2) \\ &= (5-x)(x+2) \end{aligned}$$

Note: The minus sign was taken into the first bracket to produce a neater solution.

You should now attempt questions 33 to 36 of exercise on page 41

Recall the following expansions:

$$(a+b)^2 = a^2 + 2ab + b^2$$

and $$(a-b)^2 = a^2 - 2ab + b^2$$

These can be used to factorise certain quadratics quickly.

Example Factorise (a) $4x^2 + 12x + 9$ (b) $9x^2 - 6x + 1$

Solutions (a) The term in x^2 and the constant term are both perfect squares as $4x^2 = (2x)^2$ and $9 = 3^2$.
The quadratic can be written as $(2x)^2 + 2.2x.3 + 3^2$
This is of the form $a^2 + 2ab + b^2$ where $a = 2x$ and $b = 1$
Thus $4x^2 + 12x + 9 = (2x+3)^2$

(b) The quadratic is of the form $a^2 - 2ab + b^2$ where $a = 3x$, $b = 1$
Thus $9x^2 - 6x + 1 = (3x-1)^2$

You should now attempt questions 37 to 40 of exercise 8 on page 41.

Certain quadratics can be factorised using the formula for 'the difference of two squares'

$$a^2 - b^2 = (a+b)(a-b)$$

Example Factorise $9x^2 - 25$

Solution

$$\begin{aligned} 9x^2 - 25 &= (3x)^2 - 5^2 \\ &= (3x+5)(3x-5) \end{aligned}$$

You should now attempt questions 41 to 44 of exercise 8 on page 41.

Exercise 8

Factorise the following quadratics :

(1) x^2+6x+8 (2) $x^2+9x+18$

(3) $y^2+11y+30$ (4) $z^2+20z+36$

Now return to page 35.

Factorise :

(5) $x^2-7x+10$ (6) $a^2-8a+12$

(7) $m^2-9m+14$ (8) $p^2-11p+24$

Now turn to page 36

Factorise :

(9) $x^2+2x-15$ (10) y^2+y-20

(11) z^2-z-12 (12) $w^2-3w-18$

The following examples are a mixture of the different cases shown so far.

Factorise :

(13) $a^2-10a+21$ (14) p^2-p-20

(15) $x^2+17x+60$ (16) $z^2+9z-22$

Now turn to page 37

Factorise :

(17) $2x^2+13x+15$ (18) $11y^2+17y+6$

(19) $7x^2+37x+10$ (20) $5z^2+26z+24$

Now return to page 38.

Factorise :

(21) $3x^2-8x+5$ (22) $5a^2-17a+6$

(23) $7x^2-25x+12$ (24) $11b^2-25b+6$

Now turn to page 39.

Factorise:

(25) $3x^2+x-14$ (26) $5m^2+39m-8$

(27) $3z^2-14z-5$ (28) $7a^2-23a-20$

Now return to page 39.

Factorise :

(29) $6b^2-5b-6$ (30) $12p^2+5p-3$

(31) $12x^2-25x+12$ (32) $18z^2+37z+15$

Now return to page 39.

Factorise:

(33) $4x^2-36x+80$ (34) $6a^2+21a-45$

(35) $-2x^2+11x-12$ (36) $-8b^2-2b+3$

Now return to page 40.

Factorise:

(37) $9x^2+30x+25$ (38) $16x^2-8x+1$

(39) $36x^2+12x+1$ (40) $25x^2-20x+4$

Now return to page 40.

Factorise :

(41) $16x^2-81$ (42) $64-9x^2$

(43) $8x^2-50$ (44) $108-75x^2$

Now turn to Chapter 9 on page 42.

CHAPTER 9

ALGEBRAIC FRACTIONS

Simplication of fractions

A fraction can be simplified if the numerator and denominator have factors in common.

For example, consider the expression $\dfrac{2w(x-y)}{4v(x-y)}$

The numerator and denominator have the common factor 2 and $(x-y)$. To simplify the fraction, divide the numerator and denominator by $2(x-y)$.

$$\text{Thus} \quad \frac{2w(x-y)}{4v(x-y)} = \frac{w}{2v}$$

To find common factors, it is often necessary to factorise the numerator and denominator where possible.

Example Simplify the following expressions:

(a) $\dfrac{3x^2 - 3xy}{6xy - 6y^2}$ (b) $\dfrac{x-3}{x^2 - 5x + 6}$ (c) $\dfrac{4a^2 - 9}{2a^2 - 11a + 12}$

Solutions (a)

$$\frac{3x^2 - 3xy}{6xy - 6y^2} = \frac{3x(x-y)}{6y(x-y)}$$

$$= \frac{x}{2y} \qquad \text{[dividing top and bottom by } 3(x-y)\text{]}$$

(b)

$$\frac{x-3}{x^2 - 5x + 6} = \frac{x-3}{(x-3)(x-2)}$$

$$= \frac{1}{x-2}$$

(c) Factorise the numerator using 'the difference of two squares'

$$\frac{4a^2 - 9}{2a^2 - 11a + 12} = \frac{(2a-3)(2a+3)}{(2a-3)(a-4)}$$

$$= \frac{2a+3}{a-4}$$

You should now attempt questions 1 to 9 of exercise 9 on page 49.

Example Simplify $\dfrac{\frac{1}{4}x^2 - \frac{1}{3}x}{\frac{1}{2}xy - \frac{2}{3}y}$

Solution Firstly, remove the fractions which appear in the numerator and denominator of the expression. This may be done as follows:

(i) Find the LCM of the denominators of the fractions $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ and $\frac{2}{3}$. The LCM of 4,3,2 and 3 is 12.

(ii) Multiply the numerator and denominator of the expression by 12.

Thus $\dfrac{\frac{1}{4}x^2 - \frac{1}{3}x}{\frac{1}{2}xy - \frac{2}{3}y} = \dfrac{12\left(\frac{1}{4}x^2 - \frac{1}{3}x\right)}{12\left(\frac{1}{2}xy - \frac{2}{3}y\right)}$

$$= \frac{12 \times \frac{1}{4}x^2 - 12 \times \frac{1}{3}x}{12 \times \frac{1}{2}xy - 12 \times \frac{2}{3}y}$$

$$= \frac{3x^2 - 4x}{6xy - 8y}$$

$$= \frac{x(3x-4)}{2y(3x-4)}$$

$$= \frac{x}{2y}$$

You should now attempt questions 10 to 14 of exercise 9 on page 49.

Multiplication

To multiply fractions, take the product of the numerators and the product of the denominators.

Examples

(a) $\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$

(b) $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$

(c) $x^2 \times \frac{y^3}{x} = \frac{x^2}{1} \times \frac{y^3}{x} = \frac{x^2y^3}{x} = xy^3$

(d) $\left(\frac{a}{3b}\right)^2 = \frac{a}{3b} \times \frac{a}{3b} = \frac{a^2}{9b^2}$

You should now attempt questions 15 to 24 of exercise 9 on page 49.

Examples

Simplify: (a) $\dfrac{15}{16}\times\dfrac{4}{25}$ (b) $\dfrac{18}{49}\times\dfrac{35}{27}$ (c) $\dfrac{5a^2}{8b^2}\times\dfrac{24b^5}{25a^4}$

Solutions (a) $\dfrac{15}{16}\times\dfrac{4}{25}=\dfrac{15\times 4}{16\times 25}$

Notice that the numerator and denominator have the factors 5 and 4 in common. To simplify the calculation, divide the numerator and denominator by these factors. You may find the process of cancelling helpful.

Thus $\dfrac{{}^{3}\cancel{15}\times{}^{1}\cancel{4}}{{}^{4}\cancel{16}\times{}^{5}\cancel{25}}=\dfrac{3\times 1}{4\times 5}=\dfrac{3}{20}$

(b) $\dfrac{18}{49}\times\dfrac{35}{27}=\dfrac{{}^{2}\cancel{18}\times{}^{5}\cancel{35}}{{}^{7}\cancel{49}\times{}^{3}\cancel{27}}$

$=\dfrac{2\times 5}{7\times 3}$ (dividing top and bottom by 9 and 7)

$=\dfrac{10}{21}$

(c) $\dfrac{5a^2}{8b^2}\times\dfrac{24b^5}{25a^4}=\dfrac{{}^{1}\cancel{5}\times{}^{3}\cancel{24}\times a^2\times b^5}{{}^{1}\cancel{8}\times{}^{5}\cancel{25}\times b^2\times a^4}$

$=\dfrac{1\times 3\times b^3}{1\times 5\times a^2}$

$=\dfrac{3b^3}{5a^2}$

You should now attempt questions 25 to 29 of exercise 9 on page 50.

Division

To divide by a fraction, turn it upside down and multiply.

eg.
$$\frac{3}{8} \div \frac{5}{7} = \frac{3}{8} \times \frac{7}{5}$$
$$= \frac{21}{40}$$

Examples (a)
$$\frac{4a^2}{b} \div 6a^3 = \frac{4a^2}{b} \div \frac{6a^3}{1}$$
$$= \frac{4a^2}{b} \times \frac{1}{6a^3}$$
$$= \frac{4a^2}{6a^3b}$$
$$= \frac{2}{3ab}$$

(b)
$$\frac{x^2+x-6}{3} \div \frac{x+3}{x-2} = \frac{x^2+x-6}{3} \times \frac{x-2}{x+3}$$
$$= \frac{(x^2+x-6)(x-2)}{3(x+3)}$$
$$= \frac{(x+3)(x-2)(x-2)}{3(x+3)}$$
$$= \frac{(x-2)^2}{3}$$

You should now attempt questions 30 to 36 of exercise 9 on page 50.

Addition and subtraction

To add fractions with the same denominator, simply add their numerators.

eg. $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$

and $\frac{3}{x^2+1}+\frac{x}{x^2+1}=\frac{3+x}{x^2+1}$

Subtraction of fractions with the same denominators is carried out by subtracting their denominators.

eg. $\frac{4}{2x-3}-\frac{3}{2x-3}=\frac{1}{2x-3}$

Before fractions with different denominators can be added or subtracted, they must be expressed as fractions with the same denominator. The best denominator to choose is the lowest common multiple of the denominators of the individual fractions; this is called the 'lowest common denominator'(LCD).

eg. (a) the LCD of $\frac{3}{4}$ and $\frac{5}{6}$ is 12

(b) the LCD of $\frac{4}{x+1}$ and $\frac{3}{x+2}$ is $(x+1)(x+2)$

The following examples on lowest common multiples are intended to prepare you for finding lowest denominators.

Examples Find the LCM of

(a) $x^2,\ xy,\ y^3$

(b) $6x,\ 8(x-1)^2,\ 12x(x-1)$

Solutions (a) Choose the *highest* powers of x and y appearing in the three terms and multiply them together.
The LCM is therefore x^2y^3

(b) The lowest common mutiple of the numbers 6, 8 and 12 is 24.
The highest powers of x and $(x-1)$ appearing are x and $(x-1)^2$ respectively.
The LCM is therefore $24x(x-1)^2$.

You should now attempt questions 37 to 41 of exercise 9 on page 50.

The following examples will prepare you for expressing fractions in terms of the lowest common denominator.

Examples Fill the blank in the statement:

$$\frac{4x}{3(x+2)} = \frac{\qquad}{6(x-1)(x+2)^2}$$

Solution $6(x-1)(x+2)^2$ is obtained by mutiplying the denominator $3(x+2)$ by $2(x-1)(x+2)$.

Therefore, the numerator $4x$ must also be mutiplied by $2(x-1)(x+2)$.

$$\text{Thus } \frac{4x}{3(x+2)} = \frac{4x \times 2(x-1)(x+2)}{6(x-1)(x+2)^2}$$
$$= \frac{8x(x-1)(x+2)}{6(x-1)(x+2)^2}$$

You should now attempt questions 42 to 44 of exercise 9 on page 50.

Examples Evaluate (a) $\frac{2}{9}+\frac{5}{12}$ (b) $9\frac{2}{3}-3\frac{1}{4}$

Simplify (c) $\frac{4}{xy}+3z$ (d) $\frac{4}{x-5}-\frac{3}{x}$

Solutions (a) The LCM of 9 and 12 is 36

$$\therefore \frac{2}{9}+\frac{5}{12} = \frac{8}{36}+\frac{15}{36}$$
$$= \frac{23}{36}$$

(b) The integers and fractions can be subtracted separately,

$$\text{Thus } 9\frac{2}{3}-3\frac{1}{4} = 6+\frac{2}{3}-\frac{1}{4}$$
$$= 6+\frac{8}{12}-\frac{3}{12}$$
$$= 6\frac{5}{12}$$

(c) $\frac{4}{xy}+3z=\frac{4}{xy}+\frac{3z}{1}$

$=\frac{4}{xy}+\frac{3xyz}{xy}$

$=\frac{4+3xyz}{xy}$

(d) The LCM of $x-5$ and x is $(x-5)x$

$\therefore \frac{4}{x-5}-\frac{3}{x}=\frac{4x}{(x-5)x}-\frac{3(x-5)}{(x-5)x}$

$=\frac{4x-3(x-5)}{(x-5)x}$

$=\frac{4x-3x+15}{(x-5)x}$

$=\frac{x+15}{(x-5)x}$

You should now attempt questions 45 to 57 of exercise 9 on pages 50 and 51.

Example Simplify $\frac{3}{x+2}-\frac{2x}{x^2+x-2}$

Solution Before attempting to find the LCM of the denominators, factorise the denominator x^2+x-2

Thus $\frac{3}{x+2}-\frac{2x}{x^2+x-2}=\frac{3}{x+2}-\frac{2x}{(x+2)(x-1)}$

$=\frac{3(x-1)-2x}{(x+2)(x-1)}$

$=\frac{3x-3-2x}{(x+2)(x-1)}$

$=\frac{x-3}{(x+2)(x-1)}$

You should now attempt questions 58 to 60 of exercise 9 on page 51.

(c) $$\frac{4}{xy}+3z=\frac{4}{xy}+\frac{3z}{1}$$

$$=\frac{4}{xy}+\frac{3xyz}{xy}$$

$$=\frac{4+3xyz}{xy}$$

(d) The LCM of $x-5$ and x is $(x-5)x$

$$\therefore \frac{4}{x-5}-\frac{3}{x}=\frac{4x}{(x-5)x}-\frac{3(x-5)}{(x-5)x}$$

$$=\frac{4x-3(x-5)}{(x-5)x}$$

$$=\frac{4x-3x+15}{(x-5)x}$$

$$=\frac{x+15}{(x-5)x}$$

You should now attempt questions 45 to 57 of exercise 9 on pages 50 and 51.

Example Simplify $$\frac{3}{x+2}-\frac{2x}{x^2+x-2}$$

Solution Before attempting to find the LCM of the denominators, factorise the denominator x^2+x-2

Thus $$\frac{3}{x+2}-\frac{2x}{x^2+x-2}=\frac{3}{x+2}-\frac{2x}{(x+2)(x-1)}$$

$$=\frac{3(x-1)-2x}{(x+2)(x-1)}$$

$$=\frac{3x-3-2x}{(x+2)(x-1)}$$

$$=\frac{x-3}{(x+2)(x-1)}$$

You should now attempt questions 58 to 60 of exercise 9 on page 51.

Exercise 9

All numerical exercises should be carried out without the use of a calculator.

Simplify :

(1) $\dfrac{2a+4b}{6a+12b}$ (2) $\dfrac{x^2-xy}{xy-y^2}$ (3) $\dfrac{9a^3+18ab^2}{12a^2b+24b^3}$

(4) $\dfrac{2abc-8ab}{6a^2c+24a^2}$ (5) $\dfrac{2x-3}{2x^2+5x-12}$ (6) $\dfrac{3x^2+10x-8}{3x^2+14x+8}$

(7) $\dfrac{9y^2-16}{3y^2+y-4}$ (8) $\dfrac{a^2-4b^2}{a^2-4ab+4b^2}$ (9) $\dfrac{16x^2+24xy+9y^2}{16x^2-9y^2}$

Now turn to page 43.

Simplify :

(10) $\dfrac{\frac{1}{2}xy}{\frac{1}{3}x(a-b)}$ (11) $\dfrac{\frac{1}{4}(x-5)}{x^2-25}$ (12) $\dfrac{\frac{1}{4}x^2-\frac{1}{2}x-2}{\frac{1}{3}x^2+\frac{5}{3}x+2}$

(13) $\dfrac{4x-\frac{1}{x}}{2x+3-\frac{2}{x}}$ (14) $\dfrac{9-\frac{b^2}{a^2}}{9-\frac{6b}{a}+\frac{b^2}{a^2}}$

Now return to page 43.

Simplify:

(15) $\dfrac{4}{7}\times\dfrac{5}{9}$ (16) $\dfrac{3}{5}\times\dfrac{4}{11}$ (17) $3\times\dfrac{4}{5}\times\dfrac{2}{7}$

(18) $\dfrac{4a}{b}\times\dfrac{a}{6b}$ (19) $3ab\times\dfrac{4b}{a^2}$ (20) $\dfrac{3xy}{z}\times\dfrac{yw}{2v}\times\dfrac{3zw}{xy^2}$

(21) $(x-1)^2\times\dfrac{4}{(x-1)^3}$ (22) $\dfrac{a^2+ab}{b}\times\dfrac{c}{ac+bc}$

(23) $\left(\dfrac{x}{4yz}\right)^2$ (24) $\left(\dfrac{2a^2}{3b^4}\right)^3$

Now turn to page 44.

Simplify

(25) $\frac{14}{25} \times \frac{20}{21}$

(26) $4\frac{1}{6} \times 3\frac{3}{5}$

(27) $\frac{49}{64} \times \frac{16}{25} \times \frac{20}{63}$

(28) $\frac{16a^2}{27b^3} \times \frac{81b^6}{40a^4}$

(29) $\frac{30x^4y}{11z^3} \times \frac{22z^4}{45x^2y^2}$

Now turn to page 45.

Simplify:

(30) $\frac{3}{8} \div 4$

(31) $\frac{7}{8} \div \frac{21}{40}$

(32) $3\frac{3}{5} \div 2\frac{1}{4}$

(33) $\frac{3pq}{5rs} \div \frac{p^2}{15s^2}$

(34) $x(3-x) \div \frac{3-x}{4}$

(35) $\frac{1}{a^2-b^2} \div \frac{1}{a-b}$

(36) $\frac{x^2+4x+3}{5} \div \frac{x+1}{10}$

Now turn to page 46.

Find the L.C.M. of the following :

(37) $3x\ , 4x$

(38) $3ab\ ,\ 4bc\ ,\ 6ac$

(39) $5p^2q^3, 10pq^4, 2p^2q^3$

(40) $(a-b)^2, (a-b)$

(41) $(x+3)^2, (x+3), (x-1)$

Turn to page 47.

Fill the blanks in the following :

(42) $\frac{3x}{2} = \frac{\quad}{8y}$

(43) $\frac{4}{x} = \frac{\quad}{2x^2(x-1)}$

(44) $\frac{2x-1}{x+1} = \frac{\quad}{3x^2(x+1)}$

Return to page 47.

Simplify :

(45) $\frac{3}{10} + \frac{1}{4}$

(46) $\frac{5}{8} - \frac{2}{5}$

(47) $8\frac{2}{3} + 5\frac{3}{5}$

(48) $16\frac{1}{5} - 8\frac{3}{7}$

(49) $\frac{1}{p} + \frac{1}{q}$

(50) $\frac{1}{3x} - \frac{1}{5x}$

(51) $3x + \frac{1}{x}$

(52) $\frac{a}{b} + \frac{c}{d}$

(53) $\frac{x}{3} + \frac{(x+1)}{4}$

(54) $\frac{1}{5}(x+2) - \frac{1}{3}(2x-1)$

(55) $\dfrac{4}{(y+1)^2}-\dfrac{2}{y+1}$ (56) $\dfrac{4}{x+2}-\dfrac{3}{x+3}$ (57) $\dfrac{1}{(t+2)^2}-\dfrac{2}{t+2}+\dfrac{1}{3t-1}$

Return to page 48.

Simplify :

(58) $\dfrac{5x}{x^2-x-6}-\dfrac{2}{x+2}$

(59) $\dfrac{4y}{y^2+2y+1}+\dfrac{3}{y+1}$

(60) $\dfrac{7}{z^2+3z-10}-\dfrac{2}{z^2+5z}-\dfrac{2}{z^2-2z}$

ANSWERS

Exercise 6

1. $a = \frac{F}{m}$
2. $b = \frac{A}{4a}$
3. $h = \frac{V}{\pi r^2}$
4. $x = yb$
5. $T = \frac{PV}{R}$
6. $t = \frac{q}{i}$
7. $d = \frac{ab}{cz}$
8. $a = \frac{v-u}{t}$
9. $x = \frac{v^2 - u^2}{2a}$
10. $g = \frac{v-w}{p}$
11. $u = \frac{mv - pt}{m}$
12. $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
13. $x = n(y - m)$
14. $a = \frac{2(x - ut)}{t^2}$
15. $h = \frac{A - \pi r^2}{\pi r}$
16. $d = \frac{6D - Tl}{6}$
17. $t = \frac{w - v}{av}$
18. $r = \frac{\sqrt{4p^2 - 3q^2}}{4}$
19. $w = \sqrt{\frac{T-F}{mr}}$
20. $h = \frac{av^2}{6}$
21. $l = \frac{gT^2}{4\pi^2}$
22. $g = \frac{4\pi^2 I}{MhT^2}$
23. $u = \frac{(m-n)^2}{t^2}$
24. $y = \frac{w - z^2 x}{z^2}$

Exercise 7

1. $5(x+5)$
2. $4(3x-4y)$
3. $p(a+b)$
4. $3(3a+4b-5c)$
5. $x(5x-12)$
6. $y(6y^2+7)$
7. $5x$
8. $4x^3$
9. a^2bc^2
10. $4x^2$
11. $9x(2x-3)$
12. $2pq^2(2q-3)$
13. $3x(x^2-2x+4)$
14. $4a(6b^2+2c^2-d^2)$
15. $3ab(2b^2-3c^2+bc)$
16. $P = \frac{A+5Q}{2A-3}$
17. $m = \frac{Fr}{v^2 - rg}$
18. $x = \sqrt{\frac{yz}{y+d}}$
19. $x = \frac{z^2 y}{1-z^2}$
20. $g = p\frac{(M^2+m^2)}{M^2-m^2}$

Exercise 8

(1) $(x+4)(x+2)$
(2) $(x+3)(x+6)$
(3) $(y+6)(y+5)$
(4) $(z+18)(z+2)$
(5) $(x-5)(x-2)$
(6) $(a-6)(a-2)$
(7) $(m-2)(m-7)$
(8) $(p-8)(p-3)$
(9) $(x+5)(x-3)$
(10) $(y+5)(y-4)$
(11) $(z-4)(z+3)$
(12) $(w-6)(w+3)$

(13) $(a-7)(a-3)$
(14) $(p-5)(p+4)$
(15) $(x+12)(x+5)$
(16) $(z+11)(z-2)$
(17) $(2x+3)(x+5)$
(18) $(11y+6)(y+1)$
(19) $(7x+2)(x+5)$
(20) $(5z+6)(z+4)$
(21) $(3x-5)(x-1)$
(22) $(5a-2)(a-3)$
(23) $(7x-4)(x-3)$
(24) $(11b-3)(b-2)$
(25) $(3x+7)(x-2)$
(26) $(5m-1)(m+8)$
(27) $(3z+1)(z-5)$
(28) $(7a+5)(a-4)$
(29) $(3b+2)(2b-3)$
(30) $(3p-1)(4p+3)$
(31) $(4x-3)(3x-4)$
(32) $(9z+5)(2z+3)$
(33) $4(x-5)(x-4)$
(34) $3(2a-3)(a+5)$
(35) $(2x-3)(4-x)$
(36) $(4b+3)(1-2b)$
(37) $(3x+5)^2$
(38) $(4x-1)^2$
(39) $(6x+1)^2$
(40) $(5x-2)^2$
(41) $(4x+9)(4x-9)$
(42) $(8+3x)(8-3x)$
(43) $2(2x+5)(2x-5)$
(44) $3(6+5x)(6-5x)$

Exercise 9

(1) $\frac{1}{3}$
(2) $\frac{x}{y}$
(3) $\frac{3a}{4b}$
(4) $\frac{b(c-4)}{3a(c+4)}$
(5) $\frac{1}{x+4}$
(6) $\frac{3x-2}{3x+2}$
(7) $\frac{3y-4}{y-1}$
(8) $\frac{a+2b}{a-2b}$
(9) $\frac{4x+3y}{4x-3y}$
(10) $\frac{3y}{2(a-b)}$
(11) $\frac{1}{4(x+5)}$
(12) $\frac{3(x-4)}{4(x+3)}$
(13) $\frac{2x+1}{x+2}$
(14) $\frac{3a+b}{3a-b}$
(15) $\frac{20}{63}$
(16) $\frac{12}{55}$
(17) $\frac{24}{35}$
(18) $\frac{2a^2}{3b^2}$
(19) $\frac{12b^2}{a}$
(20) $\frac{9w^2}{2v}$
(21) $\frac{4}{x-1}$
(22) $\frac{a}{b}$
(23) $\frac{x^2}{16y^2z^2}$
(24) $\frac{8a^6}{27b^{12}}$
(25) $\frac{8}{15}$
(26) 15
(27) $\frac{7}{45}$
(28) $\frac{6b^3}{5a^2}$
(29) $\frac{4x^2z}{3y}$
(30) $\frac{3}{32}$
(31) $1\frac{2}{3}$
(32) $1\frac{3}{5}$
(33) $\frac{9qs}{rp}$
(34) $4x$
(35) $\frac{1}{a+b}$
(36) $2(x+3)$
(37) $12x$
(38) $12abc$
(39) $10p^2q^4$
(40) $(a-b)^2$
(41) $(x+3)^2(x-1)$
(42) $12xy$
(43) $8x(x-1)$
(44) $3x^3(2x-1)$
(45) $\frac{11}{20}$
(46) $\frac{9}{40}$
(47) $14\frac{4}{15}$
(48) $7\frac{27}{35}$
(49) $\frac{q+p}{pq}$
(50) $\frac{2}{15x}$
(51) $\frac{3x^2+1}{x}$
(52) $\frac{ad+bc}{bd}$
(53) $\frac{7x+3}{12}$
(54) $\frac{11-7x}{15}$
(55) $\frac{2-2y}{(y+1)^2}$
(56) $\frac{x+6}{(x+2)(x+3)}$
(57) $\frac{7-3t-5t^2}{(t+2)^2(3t-1)}$
(58) $\frac{3}{x-3}$
(59) $\frac{7y+3}{(y+1)^2}$
(60) $\frac{3}{z(z+5)}$

Solving Quadratics by Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A). Solve, with the above formula, the quadratics which are in the form $ax^2 + bx + c$.
These quadratics will factorise so you can use this as a check.

1). $x^2 + 5x + 6 = 0$	2). $x^2 - 5x + 4 = 0$	3). $x^2 - 4x - 5 = 0$
4). $x^2 - 6x + 5 = 0$	5). $x^2 + 2x - 8 = 0$	6). $x^2 - 10x + 21 = 0$
7). $2x^2 + 5x + 3 = 0$	8). $2x^2 + 5x - 3 = 0$	9). $3x^2 - 4x + 1 = 0$
10). $3x^2 - 5x - 2 = 0$	11). $6x^2 + 7x + 2 = 0$	12). $3x^2 - 13x - 10 = 0$
13). $6x^2 - 11x - 10 = 0$	14). $5x^2 - 3x - 2 = 0$	15). $4x^2 + 7x - 2 = 0$
16). $6x^2 + 13x + 6 = 0$	17). $5x^2 + 6x + 1 = 0$	18). $8x^2 - 6x + 1 = 0$

B). Solve the following quadratics using the above formula to 2 decimal places.

1). $x^2 + 6x + 4 = 0$	2). $x^2 + 5x + 3 = 0$	3). $x^2 + 2x - 5 = 0$
4). $x^2 - 10x + 8 = 0$	5). $x^2 + 12x + 10 = 0$	6). $x^2 + 7x + 4 = 0$
7). $2x^2 - 5x - 4 = 0$	8). $2x^2 + 9x + 3 = 0$	9). $3x^2 + 7x + 3 = 0$
10). $2x^2 - 3x - 1 = 0$	11). $5x^2 + 9x + 2 = 0$	12). $3x^2 + 2x - 3 = 0$
13). $5x^2 + x - 2 = 0$	14). $5x^2 + 8x + 2 = 0$	15). $2x^2 - 11x + 7 = 0$
16). $3y^2 + 6y - 7 = 0$	17). $4p^2 + 7p - 6 = 0$	18). $5a^2 + 9a + 2 = 0$
19). $b^2 + 2b - 5 = 0$	20). $q^2 - 15q + 8 = 0$	21). $2t^2 - t - 4 = 0$
22). $3w^2 - 5w - 8 = 0$	23). $2x^2 + 11x + 8 = 0$	24). $3r^2 + 8r + 3 = 0$
25). $5k^2 + 9k + 2 = 0$	26). $x^2 + 3x + 1 = 0$	27). $x^2 - 2x - 4 = 0$
28). $2x^2 + 7x - 3 = 0$	29). $3x^2 - 5x - 3 = 0$	30). $5x^2 + 3x - 3 = 0$

C). Simplify and solve the following quadratics to 2 decimal places.

1). $x^2 + 10x = 5$	2). $2x^2 + 5 = 9x$	3). $2k^2 + 4k = 3$
4). $4c^2 + 9c = 3$	5). $2x^2 - x = 7$	6). $x^2 + 6 = 8x$
7). $4x^2 + 3x = 5$	8). $5x^2 = 7x - 1$	9). $3x^2 + 5 = 9x$
10). $3x^2 = 7x - 3$	11). $x^2 - 8x = 7$	12). $x^2 + 5x = 2$
13). $2x^2 = 7x + 3$	14). $4x^2 - 9x = 3$	15). $x^2 = 8x - 11$
16). $5x^2 - 2 = 6x$	17). $5 - 7x = 2x^2$	18). $14x = 3x^2 + 8$
19). $6x + 3 = 5x^2$	20). $2 = 5x^2 + 8x$	21). $12x = 3x^2 + 10$
22). $8x = 3x^2 + 2$	23). $16x = 4x^2 + 3$	24). $2x^2 = 7x - 3$

D). Simplify and solve the following quadratics to 2 decimal places.

1). $x(x - 2) = 5$	2). $2x(x + 3) + 3 = 0$	3). $(x - 2)(x - 4) = 5$
4). $(x + 1)(x - 1) = 1$	5). $(x + 1)(x - 5) = 3$	6). $(2x + 1)(x - 1) = 1$
7). $(2x - 5)(x + 3) = 8$	8). $3x(x - 1) = 5$	9). $(x + 3)^2 = 2$
10). $2x(x + 4) = 1$	11). $3x(x - 2) = 2$	12). $x(x + 8) = 2x + 1$
13). $5x(x + 3) = 3(8x - 1)$	14). $(x + 3)(x + 1) = 4$	15). $(x - 3)(2x - 5) = 2$
16). $4 = 5x(4 - 5x)$	17). $(5x - 3)(3x + 1) = 1$	18). $5x(2x - 3) = 7(3 - 2x)$
19). $x(x + 4) = 6(x + 4)$	20). $2(2x + 1)^2 + 5(2x + 1) = 1$	

Completing the Square.

A). Solve the equations.

e.g. $(f - 3)^2 = 9$, square root both sides,
$f - 3 = \pm 3$,
so $f = 3 + 3 = \underline{6}$ **or** $-3 + 3 = \underline{0}$.

1).	$x^2 = 64$	2).	$d^2 = 100$	3).	$x^2 = 49$	4).	$h^2 = 10$
5).	$(c - 2)^2 = 9$	6).	$(h - 7)^2 = 4$	7).	$(x + 3)^2 = 4$	8).	$(p + 2)^2 = 25$
9).	$(x - 1)^2 = 2$	10).	$(q + 4)^2 = 3$	11).	$(w - 3)^2 = 5$	12).	$(a + 2)^2 = 2$
13).	$(h - 6)^2 = 36$	14).	$(c - 4)^2 = 10$	15).	$(x + 3)^2 = 49$	16).	$(m - 1)^2 = 7$
17).	$(t - 8)^2 = 3$	18).	$(y + 7)^2 = 6$	19).	$(h + 9)^2 = 3$	20).	$(c + 10)^2 = 8$

B). Expand the brackets. The expressions are all perfect squares.

1).	$(x + 4)^2$	2).	$(a - 6)^2$	3).	$(r + 7)^2$	4).	$(e - 5)^2$
5).	$(p - 3)^2$	6).	$(t + 2)^2$	7).	$(b - 8)^2$	8).	$(c - 9)^2$
9).	$(k - 12)^2$	10).	$(u + 11)^2$				

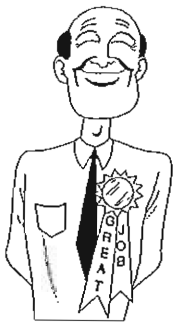
C). Add the term that will make each expression a **perfect square,** then factorise it.

e.g. To find the number we halve the coefficient of x and then square it.
$x^2 + 6x$ Halve 6 and then square it. i.e. 9.
Therefore we have $x^2 + 6x + 9 = \underline{(x + 3)^2}$

1).	$a^2 + 8a$	2).	$b^2 + 10b$	3).	$c^2 - 4c$	4).	$d^2 - 6d$
5).	$x^2 + 5x$	6).	$y^2 - 3y$	7).	$z^2 - 7z$	8).	$m^2 + 2m$
9).	$n^2 - n$	10).	$n^2 - 12n$	11).	$x^2 + 9x$	12).	$k^2 + 11k$
13).	$b^2 - 13b$	14).	$v^2 - \frac{1}{2}v$	15).	$j^2 + \frac{1}{4}j$		

D). Solve these equations by completing the square. (Some will factorise, use this as a check).

e.g. $x^2 - 8x + 3 = 0$,
$x^2 - 8x = -3$, Add 16 to both sides to complete the square.
$x^2 - 8x + 16 = 13$,
$(x - 4)^2 = 13$,
Therefore $x - 4 = \pm\sqrt{13}$,
$x = 4 \pm \sqrt{13}$, $\underline{\mathbf{x = 0.39 \text{ or } x = 7.61}}$ (2 d.p.)



1).	$a^2 + 4a - 21 = 0$	2).	$b^2 - b - 12 = 0$	3).	$n^2 + 4n + 4 = 0$
4).	$d^2 - 5d + 6 = 0$	5).	$g^2 + 5g + 4 = 0$	6).	$x^2 - 10x + 25 = 0$
7).	$q^2 + 10q + 22 = 0$	8).	$t^2 - 6t + 9 = 0$	9).	$m^2 + 6m + 7 = 0$
10).	$y^2 - 3y + 1 = 0$	11).	$d^2 + 2d - 2 = 0$	12).	$x^2 - 4x - 2 = 0$
13).	$k^2 - 5k + 2 = 0$	14).	$f^2 - f - 1 = 0$	15).	$x^2 + 3x - 2 = 0$
16).	$p^2 - 10p + 15 = 0$	17).	$v^2 + 9v + 19 = 0$	18).	$u^2 - 14u - 3 = 0$

E). Solve these equations by completing the square. (Some will factorise, use this as a check).

1).	$c^2 + 2c = 3$	2).	$d^2 - 4d = 5$	3).	$x^2 - 3x = -2$	4).	$x^2 - 2x = 4$
5).	$y^2 + 2y = 1$	6).	$h^2 + 6h = 5$	7).	$n^2 - 2m = 2$	8).	$f^2 - 4f = -1$
9).	$w^2 - 8w = -13$	10).	$d^2 + 6d = -7$	11).	$e^2 = 6e - 4$	12).	$f^2 = 11 - 4f$
13).	$j^2 = -4j - 2$	14).	$n^2 = 2n + 1$	15).	$h^2 = h + 5$	16).	$d^2 = 12d - 35$
17).	$h^2 + h = 8$	18).	$e^2 - 6e - 3 = 0$	19).	$x^2 + 5x = 15$	20).	$e^2 = 3e + 11$

Page 41. Completing the Square.

A. 1). ±8 2). ±10 3). ±7 4). ±3.16
5). 5, -1 6). 9, 5 7). -5, -1 8). 3, -7
9). -0.41, 2.41 10). -5.73, -2.27 11). 5.24, 0.76 12). -0.59, -3.41
13). 12, 0 14). 7.16, 0.84 15). 4, -10 16). 3.65, -1.65
17). 9.73, 6.27 18). -4.55, -9.45 19). -7.27, -10.73 20). -7.17, -12.83

B. 1). $x^2 + 8x + 16$ 2). $a^2 - 12a + 36$ 3). $r^2 + 14r + 49$ 4). $e^2 - 10e + 25$
5). $p^2 - 6p + 9$ 6). $t^2 + 4t + 4$ 7). $b^2 - 16b + 64$ 8). $c^2 - 18c + 81$
9). $k^2 - 24k + 144$ 10). $u^2 + 22u + 121$

C. 1). 16 2). 25 3). 4 4). 9 5). 6.25 6). 2.25
7). 12.25 8). 1 9). 1/4 10). 36 11). 20.25 12). 30.25
13). 42.25 14). 1/16 15). 1/64

D. 1). 3, -7 2). 4, -3 3). -2, -2
4). 2, 3 5). -4, -1 6). 5, 5
7). -3.27, -6.73 8). 3, 3 9). -1.59, -4.41

10). 2.62, 0.38 11). 0.73, -2.73 12). 4.45, -0.45
13). 4.56, 0.44 14). 1.62, -0.62 15). 0.56, -3.56
16). 8.16, 1.84 17). -3.38, -5.62 18). 14.21, -0.21

E. 1). 1, -3 2). 5, -1 3). 2, 1 4). 3.24, -1.24
5). 0.41, -2.41 6). 0.74, -6.74 7). 2.73, -0.73 8). 3.73, 0.27
9). 5.73, 2.27 10). -1.59, -4.41 11). 5.24, 0.76 12). 1.87, -5.87
13). -0.59, -3.41 14). 2.41, -0.41 15). 2.79, -1.79 16). 7, 5
17). 2.37, -3.37 18). 6.46, -0.46 19). 2.11, -7.11 20). 5.14, -2.14

Page 43. Solving Quadratics by Formula.

A. 1). -2, -3 2). 4, 1 3). 5, -1 4). 5, 1
5). 2, -4 6). 7, 3 7). -1, -1.5 8). 0.5, -3
9). 1, 0.33 10). 2, -0.33 11). -0.5, -0.67 12). 5, -0.67
13). 2.5, -0.67 14). 1, -0.4 15). 0.25, -2 16). -0.67, -1.5
17). -0.2, -1 18). 0.5, 0.25

B. 1). -0.76, -5.24 2). -0.7, -4.3 3). 1.45, -3.45 4). 9.12, 0.88
5). -0.9, -11.1 6). -0.63, -6.37 7). 3.14, -0.64 8). -0.36, -4.14
9). -0.57, -1.77 10). 1.78, -0.28 11). -0.26, -1.54 12). 0.72, -1.39
13). 0.54, -0.74 14). -0.31, -1.29 15). 4.77, 0.73 16). 0.83, -2.83
17). 0.63, -2.38 18). -0.26, -1.54 19). 1.45, -3.45 20). 14.45, 0.55
21). 1.69, -1.19 22). 2.67, -1.0 23). -0.86, -4.64 24). -0.45, -2.22
25). -0.26, -1.54 26). -0.38, -2.62 27). 3.24, -1.24 28). 0.39, -3.89
29). 2.14, -0.47 30). 0.53, -1.13

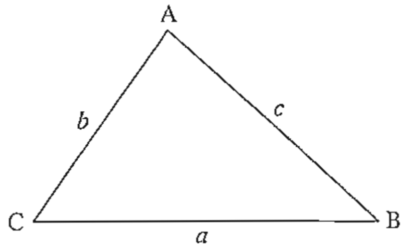
C. 1). 0.48, -10.48 2). 3.85, 0.65 3). 0.58, -2.58 4). 0.29, -2.54
5). 2.14, -1.64 6). 7.16, 0.84 7). 0.80, -1.55 8). 1.24, 0.16
9). 2.26, 0.74 10). 1.77, 0.57 11). 8.80, -0.80 12). 0.37, -5.37
13). 3.89, -0.39 14). 2.54, -0.29 15). 6.24, 1.76 16). 1.47, -0.27
17). 0.61, -4.11 18). 4, 0.67 19). 1.58, -0.38 20). 0.22, -1.82
21). 2.82, 1.18 22). 2.39, 0.28 23). 3.80, 0.20 24). 3, 0.5

D. 1). 3.45, -1.45 2). -0.63, -2.37 3). 5.45, 0.55 4). 1.41, -1.41
5). 5.46, -1.46 6). 1.28, -0.78 7). -3.15, 3.65 8). 1.88, -0.88
9). -1.59, -4.41 10). 0.12, -4.12 11). 2.29, -0.29 12). 0.16, -6.16
13). 1.36, 0.44 14). 0.24, -4.24 15). 3.78, 1.72 16). 0.4, 0.4
17). -0.4, 0.67 18). 1.5, -1.4 19). 6, -4 20). -1.84, -0.41

TRIGONOMETRY

4.8 Sine and Cosine Rules

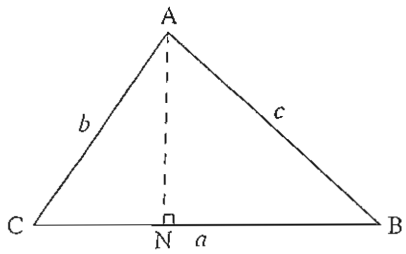
In the triangle ABC, the side opposite angle A has length a, the side opposite angle B has length b and the side opposite angle C has length c.



The *sine rule* states

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof of Sine Rule



If you construct the perpendicular from vertex A to meet side CB at N, then

$$\begin{aligned} AN &= c\sin B \quad \text{(from } \Delta \text{ ABN)} \\ &= b\sin C \quad \text{(from } \Delta \text{ ACN)} \end{aligned}$$

Hence

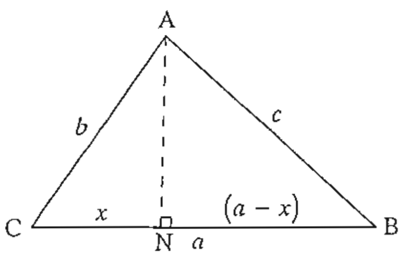
$$c\sin B = b\sin C \Rightarrow \frac{\sin B}{b} = \frac{\sin C}{c}$$

similarly for $\frac{\sin A}{a}$.

The *cosine rule* states

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

Proof of Cosine Rule



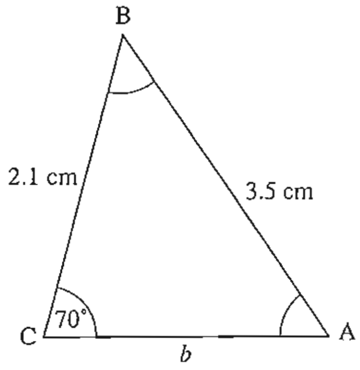
If CN = x, then NB = $a - x$ and

$$\begin{aligned} c^2 &= AN^2 + (a-x)^2 \quad \text{when } x = CN \\ &= (b\sin C)^2 + (a - b\cos C)^2, \quad \text{since } x = b\cos C \\ &= b^2\sin^2 C + b^2\cos^2 C - 2ab\cos C + a^2 \\ &= b^2(\sin^2 C + \cos^2 C) + a^2 - 2ab\cos C \end{aligned}$$

i.e. $c^2 = b^2 + a^2 - 2ab\cos C$, since $\sin^2 C + \cos^2 C = 1$

Worked Example 1

Find the unknown angles and side length of the triangle shown.



Solution

Using the sine rule,

$$\frac{\sin A}{2.1} = \frac{\sin 70°}{3.5} = \frac{\sin B}{b}$$

From the first equality,

$$\sin A = \frac{2.1 \times \sin 70°}{3.5} = 0.5638$$

$$A = 34.32°$$

Since angles in a triangle add up to 180°,

$$B = 180° - 70° - A = 75.68°$$

From the sine rule,

$$\frac{\sin 70°}{3.5} = \frac{\sin B}{b}$$

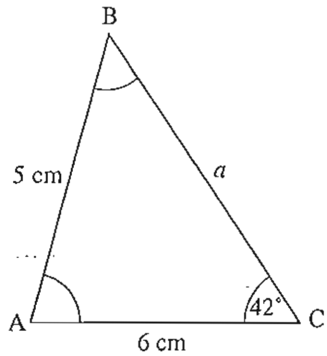
$$b = \frac{3.5 \times \sin B}{\sin 70°}$$

$$= \frac{3.5 \times \sin 75.68°}{\sin 70°}$$

$$= 3.61 \text{ cm}$$

Worked Example 2

Find two solutions for the unknown angles and side of the triangle shown.



Solution

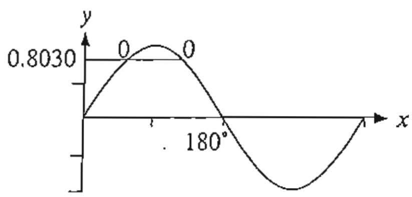
Using the sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{6} = \frac{\sin 42°}{5}$$

From the second equality,

$$\sin B = \frac{6 \times \sin 42°}{5} = 0.8030$$

A graph of $\sin x$ shows that between 0° and 180° there are two solutions for B.



These solutions are B = 53.41° and, by symmetry, B = 180 − 53.41
= 126.59°

Solving for angle A we have

$$A = 180° - 42° - B$$

when B = 126.59°, A = 11.41°

when B = 53.41°, A = 84.59°

From the sine rule,

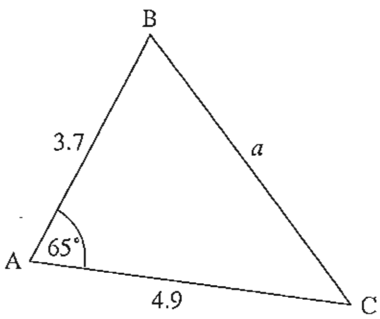
$$a = \frac{6 \times \sin A}{\sin B}$$

For A = 84.59°, B = 53.41°, a = 7.44 cm

For A = 11.41°, B = 126.59°, a = 1.48 cm

Worked Example 3

Find the unknown side and angles of the triangle shown.



Solution

To find a, use the cosine rule:

$$a^2 = 3.7^2 + 4.9^2 - 2 \times 3.7 \times 4.9 \times \cos 65°$$

$$a^2 = 22.3759$$

$$a = 4.73 \quad \text{(to 2 d.p.)}$$

To find the angles, use the sine rule:

$$\frac{\sin 65°}{a} = \frac{\sin B}{4.9} = \frac{\sin C}{3.7}$$

$$\sin B = \frac{4.9 \times \sin 65°}{a} = \frac{4.9 \times \sin 65°}{4.73} = 0.9389$$

$$B = 69.86°$$

$$\sin C = \frac{3.7 \times \sin 65°}{a} = \frac{3.7 \times \sin 65°}{4.73} = 0.7090$$

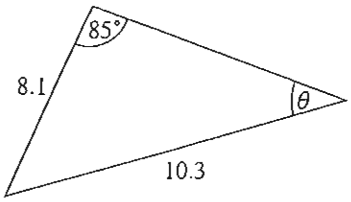
$$C = 45.15°$$ (alternatively, use A + B + C = 180° to find C)

Checking, A + B + C = 65° + 69.86° + 45.15° = 180.01°. The three angles should add to 180°; the extra 0.01° is due to rounding errors.

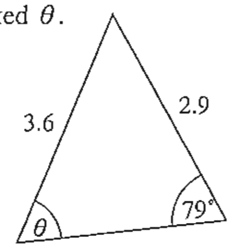
Exercises

1. For each of the triangles, find the unknown angle marked θ.

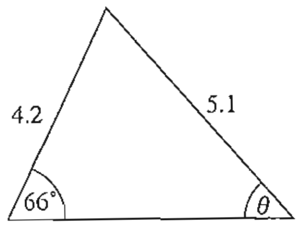
(a)



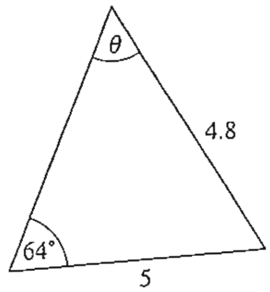
(b)



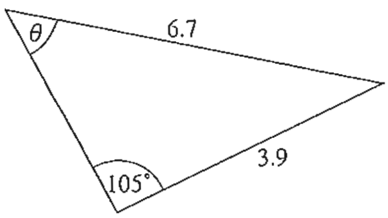
(c)



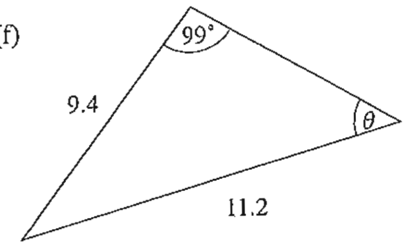
(d)



(e)

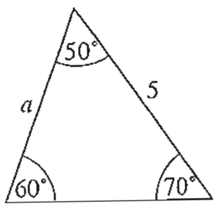


(f)

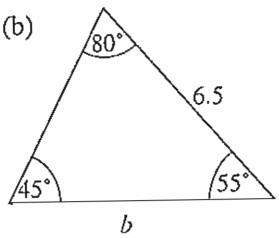


2. For each triangle, find the unknown side marked a, b or c.

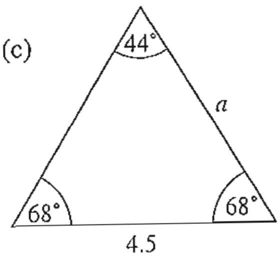
(a)



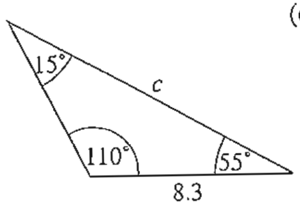
(b)



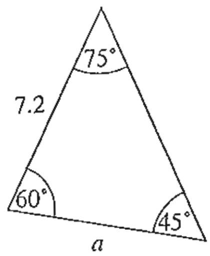
(c)



(d)



(e)



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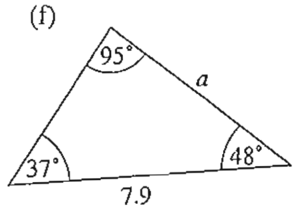
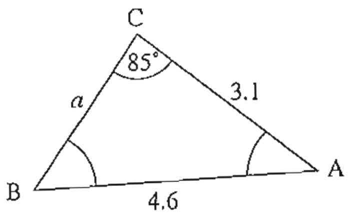
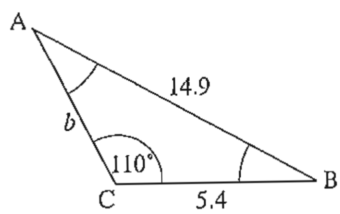


3. For each of the triangles, find the unknown angles and sides.

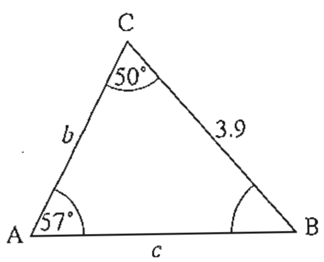
(a)



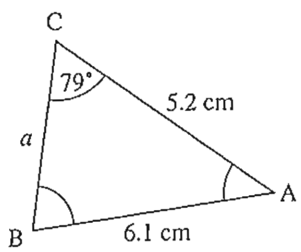
(b)



(c)

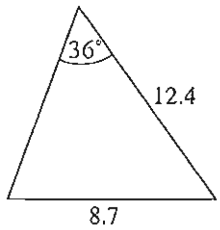


(d)

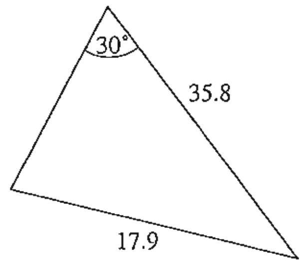


4. Which of the following triangles could have *two* solutions?

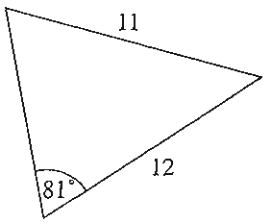
(a)



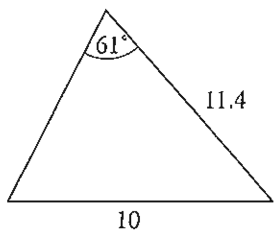
(b)



(c)



(d)

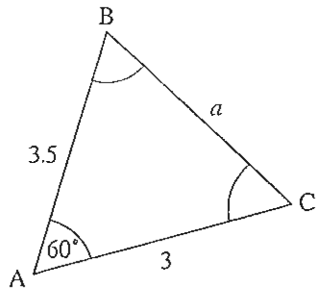


5. Find the remaining angles and sides of the triangle ABC if A = 67°, a = 125 and c = 100.

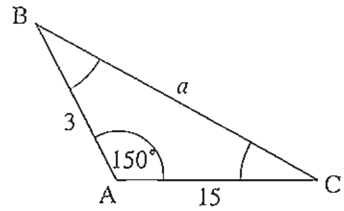
6. Find the remaining angles and sides of the triangle ABC if B = 81°, b = 12 and c = 11.

7. For each of the following triangles, find the unknown angles and sides.

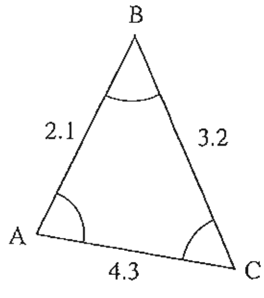
(a)



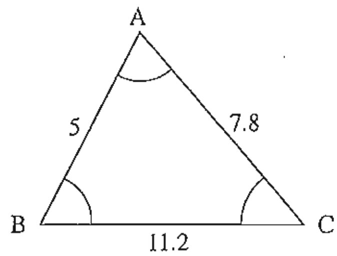
(b)



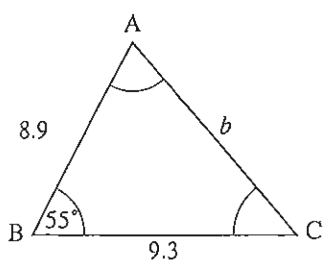
(c)



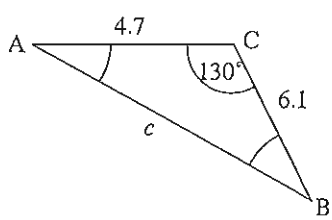
(d)



(e)



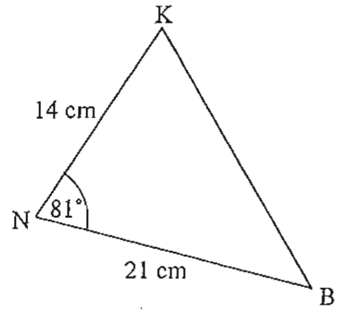
(f)



12. (a) Calculate the length KB.

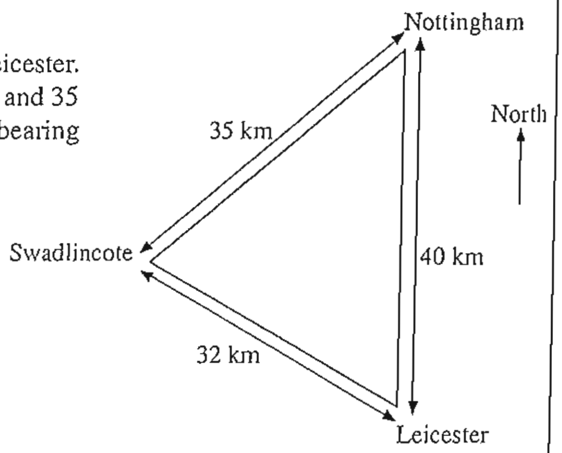
(b) Calculate the size of the angle NKB.

(LON)



13. Nottingham is 40 km due north of Leicester. Swadlincote is 32 km from Leicester and 35 km from Nottingham. Calculate the bearing of Swadlincote from Leicester.

(MEG)

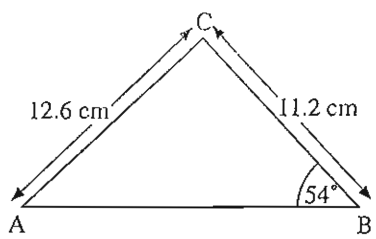


14. In triangle ABC, AC = 12.6 cm, BC = 11.2 cm and angle B = 54°. The lengths AC and BC are correct to the nearest millimetre and angle B is correct to the nearest degree. Use the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

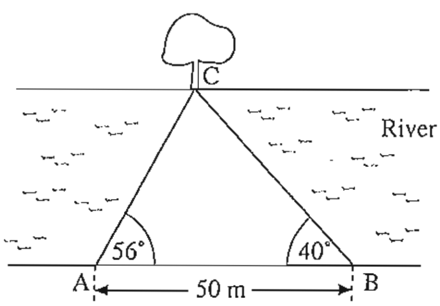
to calculate the smallest possible value of angle A.

(MEG)

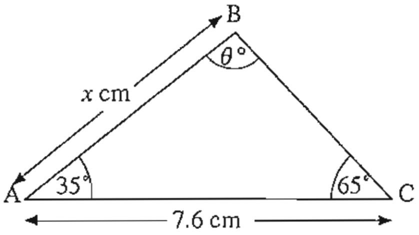


15. The banks of a river are straight and parallel. To find the width of the river, two points, A and B, are chosen 50 metres apart. The angles made with a tree at C on the opposite bank are measured as angle CAB = 56°, angle CBA = 40°. Calculate the width of the river.

(SEG)



16. In triangle ABC, AC = 7.6 cm, angle BAC = 35°, angle ACB = 65°. The length of AB is x cm. The size of angle ABC is θ°.



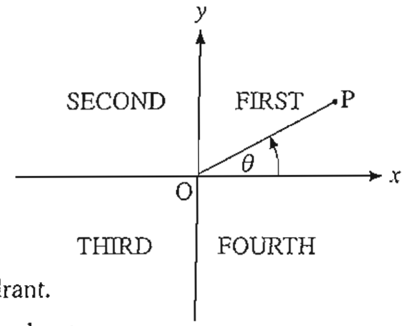
(a) Write down the value of θ.

(b) Hence calculate the value of x.

(NEAB)

4.9 Angles Larger than 90°

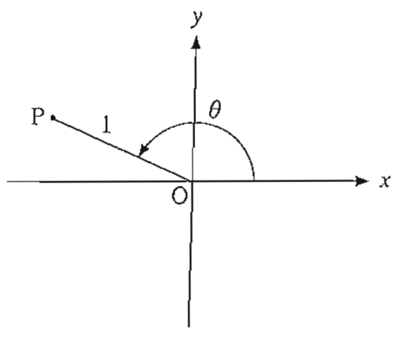
The x - y plane is divided into four quadrants by the x and y axes. The angle θ that a line OP makes with the positive x-axis lies between 0° and 360°.



Angles between 0 and 90° are in the *first* quadrant.
Angles between 90° and 180° are in the *second* quadrant.
Angles between 180° and 270° are in the *third* quadrant.
Angles between 270° and 360° are in the *fourth* quadrant.

Angles bigger than 360° can be reduced to lie between 0° and 360° by subtracting multiples of 360°.

The trigonometric formulae, $\cos\theta$ and $\sin\theta$, are defined for all angles between 0° and 360° as the coordinates of a point, P, where OP is a line of length 1, making an angle θ with the positive x-axis.



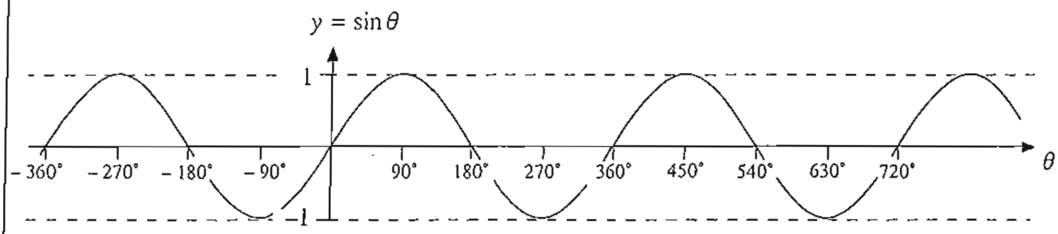
Information

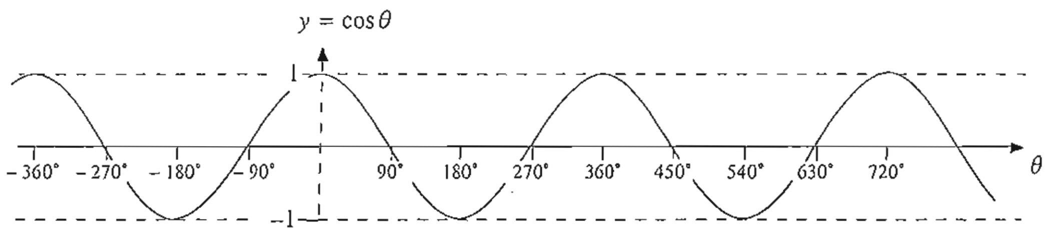
The Greeks, (in their analysis of the arcs of circles) were the first to establish the relationships or ratios between the sides and the angles of a right angled triangle.
The Chinese also recognised the ratios of sides in a right angled triangle and some survey problems involving such ratios were quoted in Zhou Bi Suan Jing.
It is interesting to note that sound waves are related to the sine curve. This discovery by Joseph Fourier, a French mathematician, is the essence of the electronic musical instrument developments today.

Some important values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are shown in this table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	infinite

The graphs of $\sin\theta$ and $\cos\theta$ for any angle are shown in the following diagrams.





The graphs are examples of *periodic functions*. Each basic pattern repeats itself every 360°. We say that the *period* is 360°.

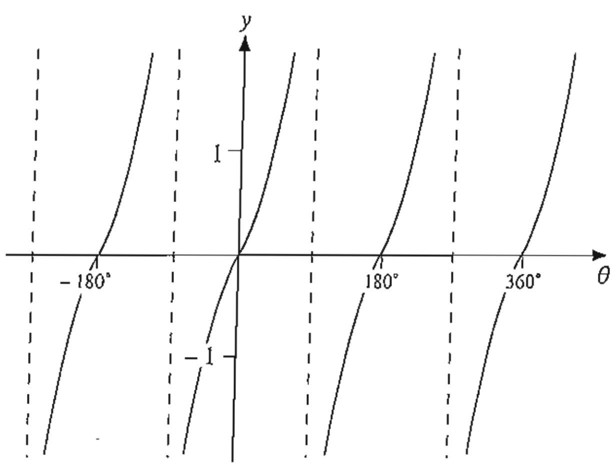
Sin θ and $\cos\theta$ are often called *sinusoidal functions*.

Note

For any angle, note that $\sin(\theta - 90°) = \cos\theta$.

The graph of $\tan\theta$ has period 180°.

It is an example of a *discontinuous graph.*



The trigonometric equations $\sin\theta = a$, $\cos\theta = b$ and $\tan\theta = c$ can have many solutions. The inverse trigonometric keys on a calculator ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$), give the *principal value solution.*

For $\sin\theta = a$ and $\tan\theta = c$, the principal value solution is in the range $-90° < \theta \le 90°$.

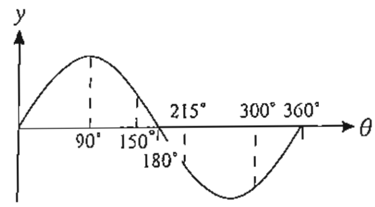
For $\cos\theta = b$, the principal value solution is in the range $0 \le \theta < 180°$.

Worked Example 1

Sketch a graph of $\sin\theta$ for $0 \le \theta \le 360°$. From the graph, deduce the values of $\sin 150°$, $\sin 215°$, $\sin 300°$.

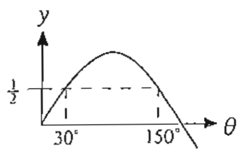
Solution

A sketch of the graph of $\sin\theta$ looks like this.



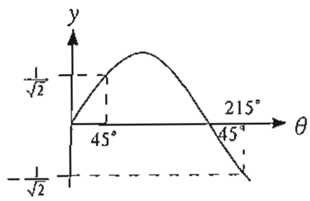
From the symmetry of the curve we can deduce that

$$\sin 150° = \sin 30° = \frac{1}{2}$$

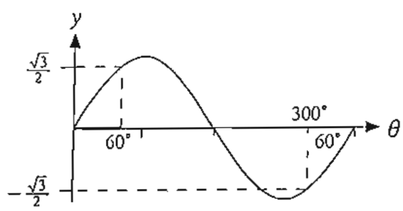


$$\sin 180° = 0$$

$$\sin 215° = -\sin 45° = -\frac{1}{\sqrt{2}}$$



$$\sin 300° = -\sin 60° = -\frac{\sqrt{3}}{2}$$



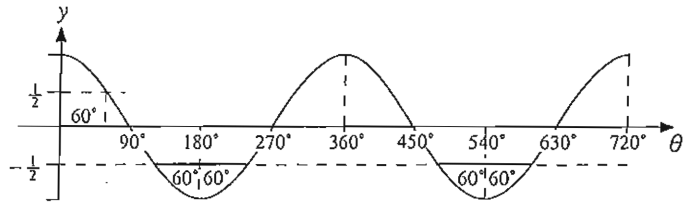
Worked Example 3

If $\cos\theta = -\frac{1}{2}$, how many values of the angle θ are possible for $0 \le \theta \le 720°$?

Find these values for θ.

Solution

A graph of $\cos\theta$ shows that there are four possible values for θ.



Using the symmetry of the graph, the values of θ are

$$\theta = 120°,\ 240°,\ 480°,\ 600°$$

The solution in the range $0 \le \theta < 180°$, $\theta = 120°$, is called the *principal value.*

Worked Example 4

Use a calculator to solve the equation $\sin\theta = -0.2$.

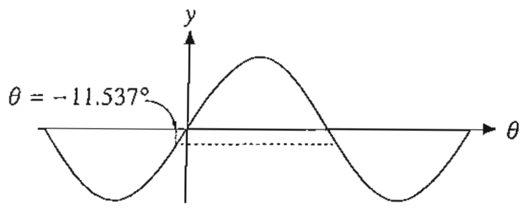
Sketch the sine graph to show this solution. Give the principal value solution.

Solution

Using the $\sin^{-1}$ key on a calculator gives

$$\theta = \sin^{-1}(-0.2) = -11.537°$$

A sketch of the graph of $\sin\theta$ shows why θ is negative.



The principal value solution is $-11.537°$.

Worked Example 5

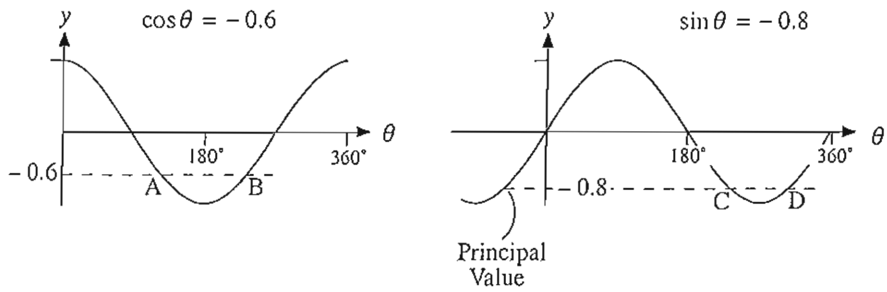
An angle θ is such that $\cos\theta = -0.6$, $\sin\theta = -0.8$ and $0 \le \theta \le 360°$.
Deduce in which quadrant the angle lies,

(a) from the graphs of $\sin\theta$ and $\cos\theta$, and

Hence, using a calculator, find the value of θ.

Solution

(a) The following graphs show the possible solutions for θ between 0° and 360°.



From the graphs we deduce that the value of θ for which $\cos\theta = -0.6$ and $\sin\theta = -0.8$ must lie between 180° and 270°, i.e. at point B on the cosine curve and at point C on the sine curve.

The $\cos^{-1}$ and $\sin^{-1}$ keys on a calculator give the principal values

$$\theta = \cos^{-1}(-0.6) = 126.87°$$

$$\theta = \sin^{-1}(-0.8) = -53.13°$$

From the graph of $\sin\theta$, for point C we deduce that

$$\theta = 180° + 53.13°$$
$$= 233.13°$$

Exercises

2. Sketch graphs of $\sin\theta$ and $\cos\theta$ for $0 \le \theta \le 720$. Without using a calculator, use the symmetry of the graphs to find the values of the $\sin\theta$ and $\cos\theta$ in problem 1.

   Now check your answers with a calculator.

3. Use a calculator to find the values of the following. In each case show the answer on sketch graphs of $\sin\theta$ or $\cos\theta$.

(a)	$\sin 130°$	(b)	$\sin 235°$	(c)	$\sin 310°$
(d)	$\sin 400°$	(e)	$\sin 830°$	(f)	$\sin 1310°$
(g)	$\cos 170°$	(h)	$\cos 190°$	(i)	$\cos 255°$
(j)	$\cos 350°$	(k)	$\cos 765°$	(l)	$\cos 940°$

4. Sketch a graph of $y = \sin\theta$ for $-360° \le \theta \le 720°$. For this domain of θ, how many solutions are there of the equation $\sin\theta = -\frac{1}{\sqrt{2}}$?

   Use the symmetry of the graph to deduce these solutions. What is the principal value?

5. Sketch a graph of $y = \cos\theta$ for $-360° \le \theta \le 720°$. For this domain of θ, how many solutions are there of the equation $\cos\theta = \frac{1}{2}$?

   Use the symmetry of the graph to deduce these solutions. What is the principal value?

6. Using a calculator and sketch graphs, find all the solutions of the following equations for $-360 \le \theta \le 360°$.

(a)	$\sin\theta = 0.7$	(b)	$\sin\theta = -0.4$	(c)	$\sin\theta = -1$
(d)	$\cos\theta = 0.6$	(e)	$\cos\theta = -0.4$	(f)	$\cos\theta = -1$

7. Use a calculator and a sketch graph of $y = \tan\theta$ to solve the equation for $0 \le \theta \le 720°$.

(a)	$\tan\theta = 0.25$	(b)	$\tan\theta = 1$	(c)	$\tan\theta = -0.5$

8. In each of the following problems find the value of θ in the range 0 to 360° that satisfies both equations.

   (a) $\cos\theta = 0.6$ and $\sin\theta = -0.8$

   (b) $\cos\theta = -0.8$ and $\sin\theta = 0.6$

   (c) $\sin\theta = -0.6428$ and $\cos\theta = -0.7660$ (each correct to 4 d.p.)

   (d) $\sin\theta = -1$ and $\cos\theta = 0$

9. Use a graphic calculator or computer software for this problem.

   (a) Draw a graph of $y = \sin 2x$ for values of x between $-360°$ and $360°$.

   (b) Compare your graph with $y = \sin x$. What is the period of the function $\sin 2x$?

   (c) Repeat parts (a) and (b) for $y = \sin 3x$ and $y = \sin \frac{1}{2}x$.

   (d) Use your answers to sketch a graph of $y = \sin ax$.

   (e) Draw a graph of $y = 2\sin x$ for values of x between $-360°$ and $360°$. What is the relationship between the graphs of $y = 2\sin x$ and $\sin x$?

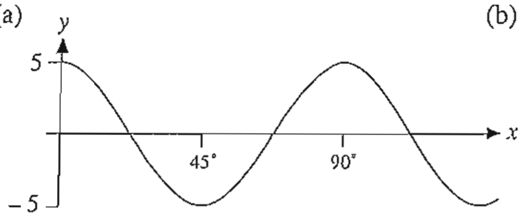
   (f) Repeat part (e) for $y = 3\sin x$ and $\frac{1}{2}\sin x$.

   (g) Use your answers in parts (e) and (f) to sketch a graph of $b\sin x$.

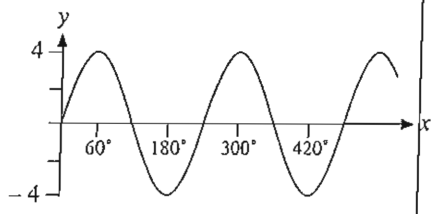
   (h) Sketch a graph of $y = b\sin ax$.

10. Find formulae in terms of sine or cosine for the following graphs.

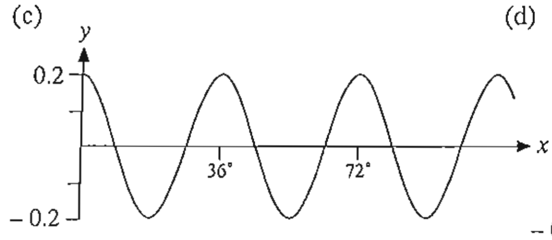
(a)



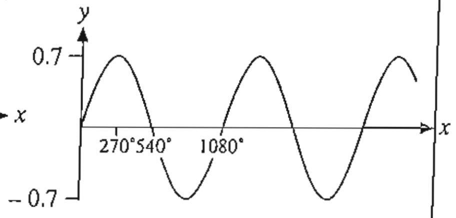
(b)



(c)



(d)



11. Draw graphs of the following.

    (a) $y = 1 + \cos x$ (b) $y = 3 + \cos x$ (c) $y = \cos x - 2$

    What is the relationship between these graphs and the graph of $y = \cos x$?

12. At a time t hours after midnight, the depth of water, d, in metres, in a harbour is given by

$$d = 8 + 5\sin(30t)^\circ$$

    Draw up a table to show the depth of water in the harbour on each hour of the day.

13. The mean monthly temperature in Crapstone, Devon, in August is 21°C and the minimum temperature in February is 0°C. Assuming that the variation in temperature is periodic satisfying a sine function, obtain a mathematical model to represent the mean monthly temperature. Use your model to predict the mean monthly temperature in June and January.

14. The variation in body temperature is an example of a biological process that repeats itself approximately every 24 hours, and is called a *circadian rhythm*. Body temperature is highest (98.9°C) around 5 pm (1700 hours) and lowest (98.3°C) around 5 am (0500 hours). Let T be the body temperature in °C and t be the time in hours.

    (a) Sketch a curve of the body temperature against time, using the given information.

    (b) Choosing $t = 0$ so that the model of temperature is a cosine function, find a formula of the form that fits the given information.

15. This question is about angles between 0° and 360°.

    (a) Find the *two* solutions of the equation

$$\cos x = 0.5$$

    (b) Angle p satisfies the equation

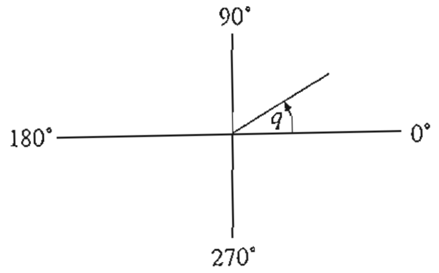
$$\sin p = \sin 210^\circ$$

    Angle p is not equal to 210°.

    Find the value of p.

    (c) Sketch the graph of $y = 5\sin x$.

(d) Angle q is shown in the diagram.



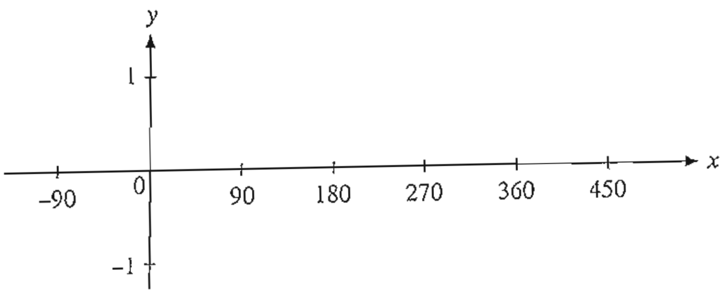
Angles q and r are connected by the equation

$$\tan q = \tan r$$

Copy the diagram and mark clearly the angle r.

(SEG)

16. (a) Sketch the graphs of $y = \cos x°$ on axes similar to those below.



(b) Use your calculator to find the value of x between 0 and 90 for which $\cos x° = 0.5$.

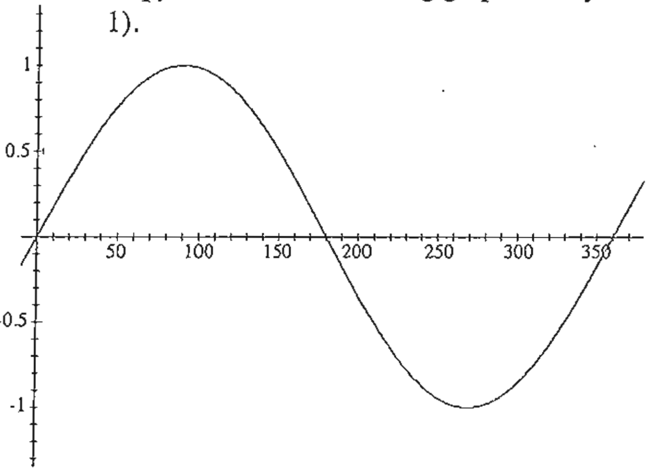
(c) Using your graph and the answer to part (b), find two more solutions in the range $-90 \le x \le 450$ for which $\cos x° = 0.5$.

(MEG)

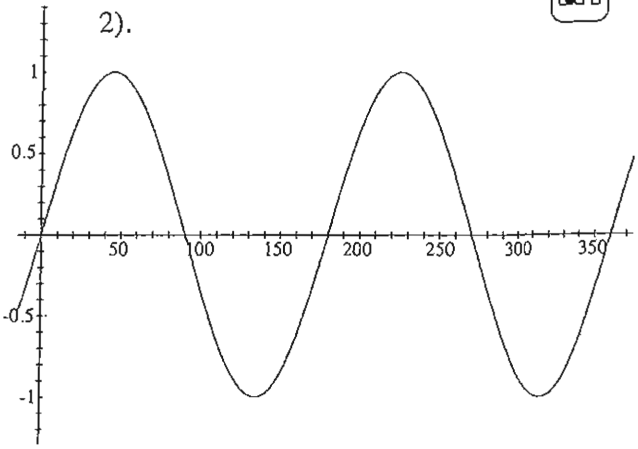
Trigonometric Graphs 2.

Copy each of the following graphs into your books and state the function.

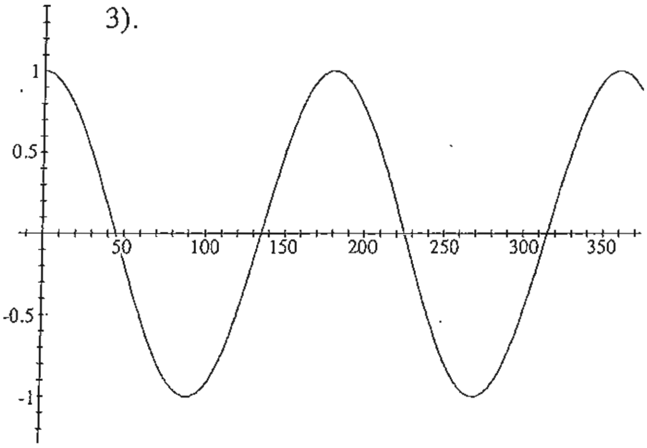
1).



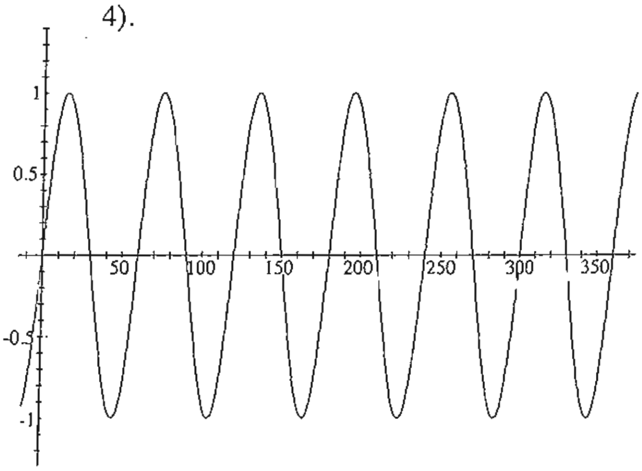
2).



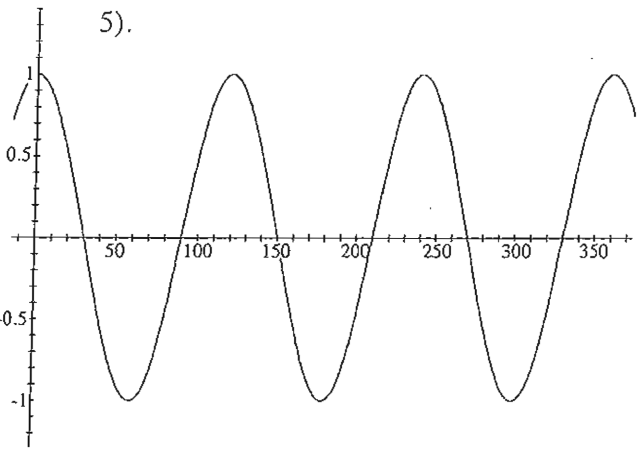
3).



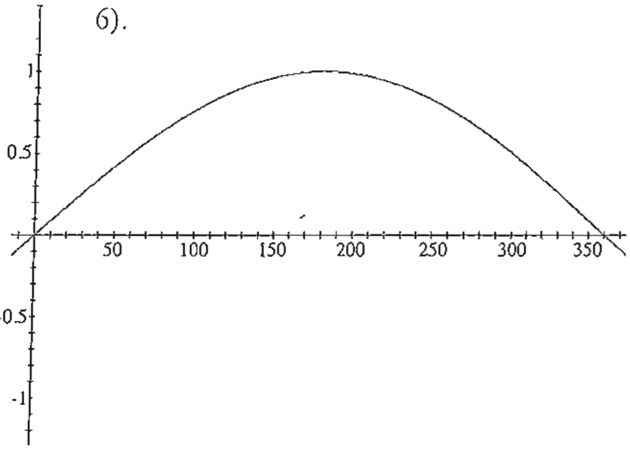
4).



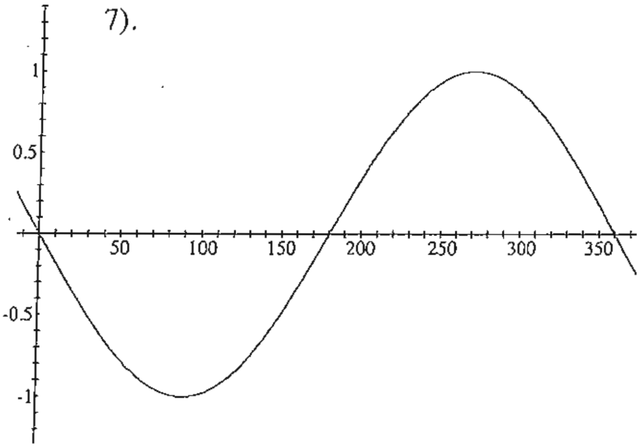
5).



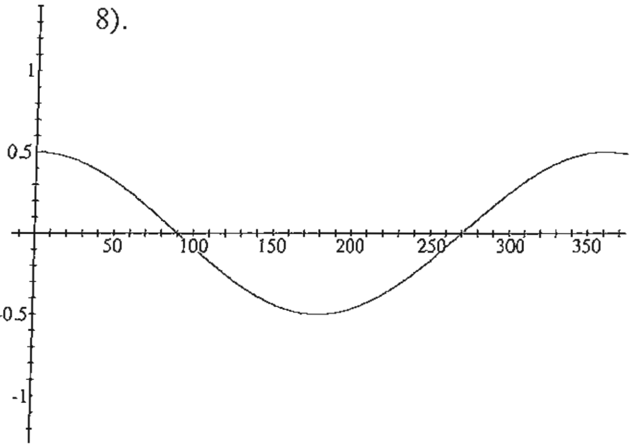
6).



7).

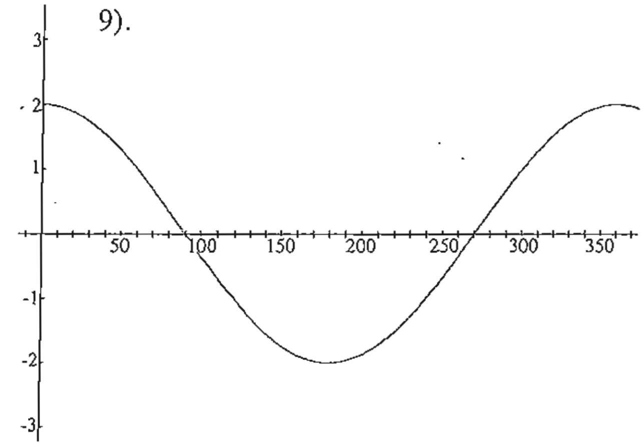


8).

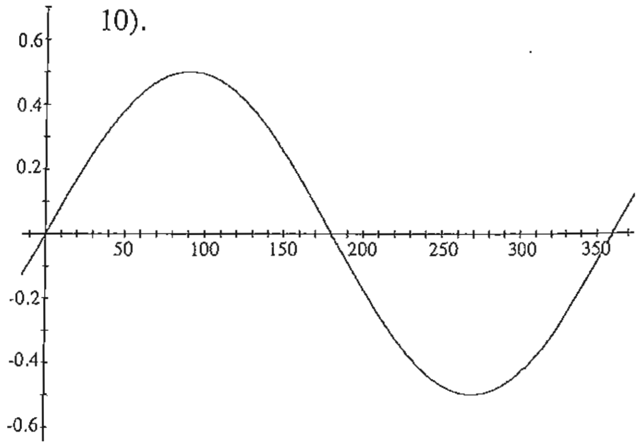


Read the scales!

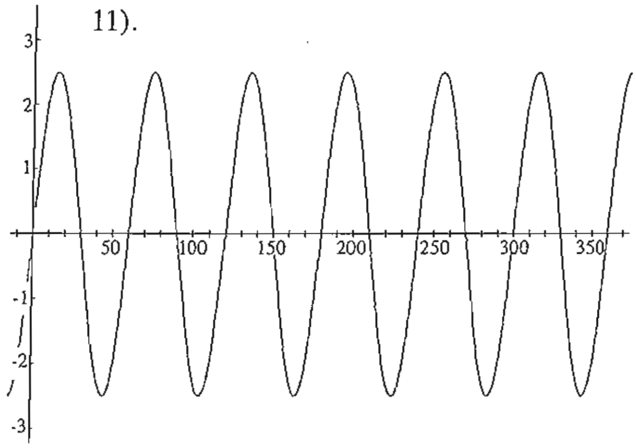
9).



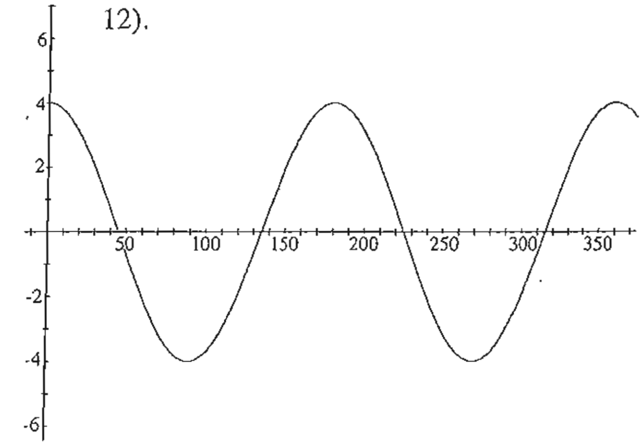
10).



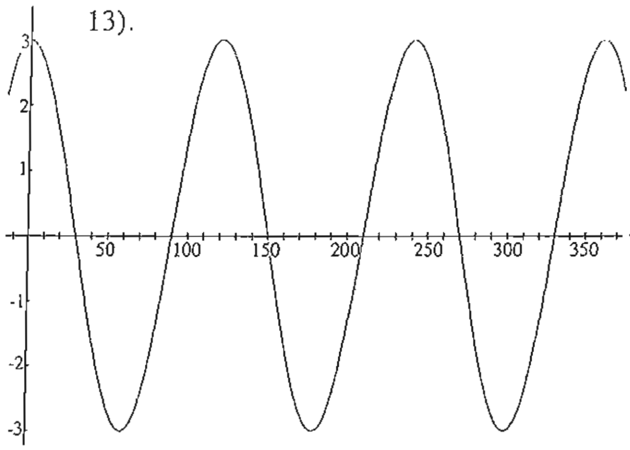
11).



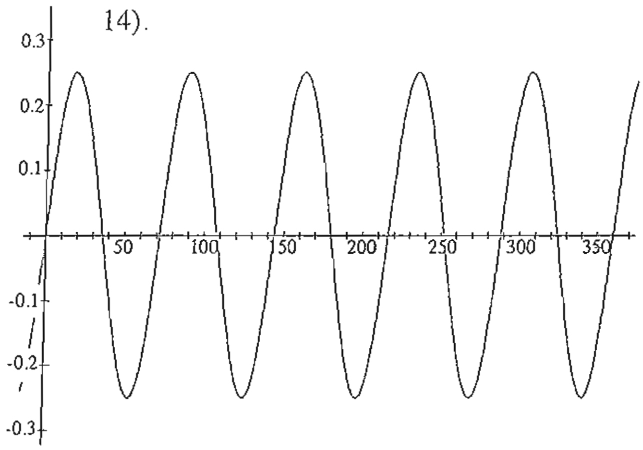
12).



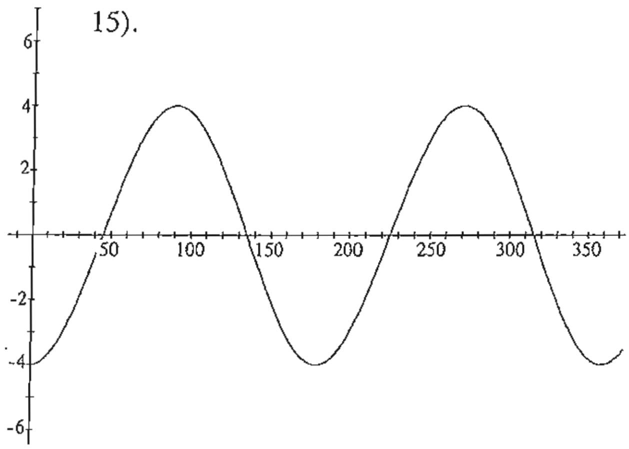
13).



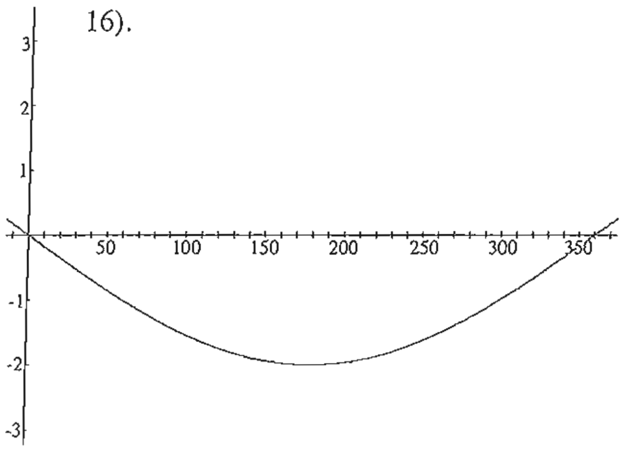
14).



15).



16).



4.8 Sine and Cosine Rules

1. (a) 51.6° (b) 52.3° (c) 48.8° (d) 69.4° (e) 34.2° (f) 56.0°
2. (a) 5.43 (b) 9.05 (c) 6.01 (d) 30.13 (e) 9.84 (f) 4.77

3. (a) A = 52.8° B = 42.2° a = 3.68
   (b) A = 19.9° B = 50.1° b = 12.16
   (c) B = 73° b = 4.45 c = 3.56
   (d) A = 44.2° B = 56.8° a = 4.33

4. (a) Yes (b) No, only one (c) No, impossible even for one (d) Yes

5. B = 65.6°, C = 47.4°, b = 123.6

6. A = 34.1°, C = 64.9°, a = 6.25

7. (a) B = 52.4°, C = 67.6°, a = 3.28 (b) B = 25.1°, C = 4.9°, a = 17.66

   (c) A = 45.5°, B = 106.6°, C = 27.9°

   (d) A = 120.5°, B = 36.9°, C = 22.6°

   (e) b = 8.41, A = 64.92°, C = 60.08°

   (f) c = 9.81, A = 28.45°, B = 21.55°

8. (a) 263.7 m (b) 192.9 m

9. 2.65 miles

10. (a) 117.3° (b) 10.2 m

11. 47.96°

12. (a) 23.35 cm (b) 62.66°

13. 303.1°

14. 45.98 (if no allowance made for inaccurate measurements) or 46.87

15. 26.8 m

16. (a) 80° (b) 6.99 cm

4.9 Angles Larger than 90°

1. (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{2}$ (e) $-\frac{1}{\sqrt{2}}$ (f) -1

   (g) $\frac{1}{2}$ (h) $\frac{\sqrt{3}}{2}$ (i) $\frac{1}{\sqrt{2}}$ (j) $-\frac{\sqrt{3}}{2}$ (k) -1 (l) $-\frac{1}{2}$

3. (a) 0.766 (b) -0.819 (c) -0.766 (d) 0.643 (e) 0.940
   (f) -0.766 (g) -0.985 (h) -0.985 (i) -0.259 (j) 0.985
   (k) 0.707 (l) -0.766

4. 6 ; −135°, −45°, 225°, 315°, 585°, 675°

5. 6 ; −300°, −60°, 60°, 300°, 420°, 660°

6. (a) −315.6°, −224.4°, 44.4°, 135.6°
 (b) −156.4°, −23.6°, 203.6°, 336.4° (c) −90°, 270°
 (d) −306.9°, −53.1°, 53.1°, 306.9°
 (e) −246.4°, −113.6°, 113.6°, 246.4° (f) −180°, 180°

7. (a) 14.0°, 194.0°, 374.0°, 554.0° (b) 45°, 225°, 405°, 585°
 (c) 153.4°, 333.4°, 513.4°, 693.4°

8. (a) 306.9° (b) 143.1° (c) 220° (d) 270°

9. (b) 180° (c) 120°, 720°

10. (a) $y = 5\cos 4x$ (b) $y = 4\sin\left(\frac{3x}{2}\right)$ (c) $y = 0.2\cos 10x$

 (d) $y = 0.7\sin\left(\frac{x}{3}\right)$

13. $y = 10.5\sin 30(x-4) + 10.5$; 19.6 °C, 1.4 °C

14. (b) $T = 98.6 + 0.3\cos 15(t-17)$

15. (a) 60°, 300° (b) 330° (d) $r = q + 180°$

16. (b) 60° (c) 300°, 420°

Page [redacted]. Trigonometric Graphs 2.

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1).	$y = \sin\theta$	2).	$y = \sin 2\theta$	3).	$y = \cos 2\theta$	4).	$y = \sin 6\theta$
5).	$y = \cos 3\theta$	6).	$y = \sin \frac{1}{2}\theta$	7).	$y = -\sin\theta$	8).	$y = \frac{1}{2}\cos\theta$

Page [redacted].

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9).	$y = 2\cos\theta$	10).	$y = 0.5\sin\theta$	11).	$y = 2.5\sin 6\theta$	12).	$y = 4\cos 2\theta$
13).	$y = 3\cos 3\theta$	14).	$y = \frac{1}{4}\sin 5\theta$	15).	$y = -4\cos 2\theta$	16).	$y = -2\sin \frac{1}{2}\theta$